د ریاضي دریم کتاب

Hva skal jeg velge?

ليکوال : عبدالله (وردک)

بِسْمِ اللهِ الرَّحْمٰنِ الرَّحِيْمِ

"In the name of Allah ,the Most Beneficent ,the Most Merciful."

شروع کوم په نامه د الله(ج) چه رحمن او رحیم دی

Narrated Anas (RA): The Prophet(SAW) used to say, "O Allah! Our Lord!

Give us in this world that, which is good and in the Hereafter that,

which is good and save us from the torment of the Fire" (Al-Bukhari)

ګرانو او محترمو ورونو اوخویندو السلام علیکم ورحمت الله وبرکاته !

له الله(ج) څخه هیله کوم چه تاسې ټول روغ او جوړ وی٠

دغه لاندي د ریاضي درسونه ما د انګلستان په یوه ښار کې چي سوتهمپټن
نومبري، د افغانانو اولادو ته خاص د ثواب په خاطر درس ورکاوه٠
که چیرې دغه درسونه په صحیح توګه پکار واچول شي، نو دغو لاندې
شاګردانو ته به ان شاءالله ګټور واقع شي

الف ـ هغو شاګردانو ته چې د کالج زده کوونکي وي **(A-Level)**

ب ـ د پوهنتون زده کوونکي هم ګټه تري اخستی شي

Dear Brothers and Sisters! السلام علیکم و رحمت الله و برکاته

I kindly request each one of you to remember me, my parents, my
children, my family and the entire Muslims of the world in your daily Prayers.

ګرانو او محترمو ورونو او خویندو السلام علیکم و رحمت الله و برکاته!

له الله(ج) څخه هیله کوم چې تاسې ټول روغ او جوړ وی او الله(ج) د وکړي
چه ماته ، زما مور او پلار ته ، زما اولادونو ته ، زما ټولې کورنۍ ته او
د نړۍ ټولو مسلمانانو ته د زړه له کومي دعا وکړی٠ والسلام

نوټ: دغه درسي لکچر نوټونه دي، او باید د کتاب په سترګه ورته ونه کتل شي٠

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PAPER-1

Ex-1-1: **(i)** Evaluate $\left(\dfrac{1}{8}\right)^{-\frac{2}{3}} = \dfrac{(1)^{-\frac{2}{3}}}{(8)^{-\frac{2}{3}}} = \dfrac{(8)^{\frac{2}{3}}}{(1)^{\frac{2}{3}}} = \dfrac{(8)^{\frac{2}{3}}}{1} = \left(2^3\right)^{\frac{2}{3}} = 2^{3\times\frac{2}{3}} = 2^2 = 4$

(ii) Simplify $\dfrac{\left(2a^3c\right)^3}{32a^6c^9} = \dfrac{2^3a^9c^3}{32a^6c^9} = \dfrac{8a^9c^3}{32a^6c^9} = \dfrac{a^3}{4c^6}$

Ex-1-2: **A** is the point (3, - 4) and **B** is the point (- 7, 6). **M** is the midpoint of **AB**. Determine whether the line with equation $y = -3x - 5$ passes through **M**.

$$M\left(\dfrac{-7+3}{2}, \dfrac{6-4}{2}\right) = M\left(\dfrac{-4}{2}, \dfrac{2}{2}\right) = M\left(-2, 1\right)$$

$$y = -3x - 5$$

$$1 \overset{?}{=} -3(-2) - 5$$

$$1 \overset{?}{=} 6 - 5$$

$$1 \overset{?}{=} 1 \quad \Rightarrow ok$$

Ex-1-3: Fig-1-3 shows the graph of **y = f(x)**.

$$y = f(x) = x^2$$

Fig-1-3

Draw the graphs of the following.

(i) $y = f(x) - 2$

(ii) $y = f(x-3)$

Ans:

Fig-1-3

Graph showing $y = f(x) = x^2$, $y = f(x-3) = (x-3)^2$ with point $3,0$, and $y = f(x) - 2 = x^2 - 2$ with point $0,-2$.
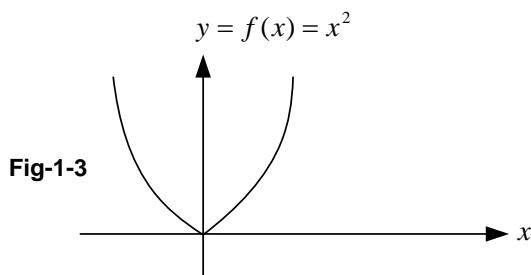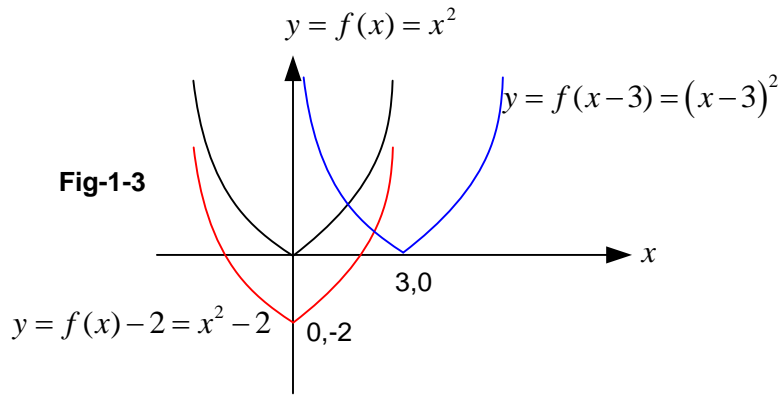
Ex-1-4: **(i)** Expand and simplify $\left(2\sqrt{3} - 3\sqrt{2}\right)^2$

Ans:

$$\left(2\sqrt{3} - 3\sqrt{2}\right)^2 = \left(2\sqrt{3}\right)^2 - 2\left(2\sqrt{3}\right)\left(3\sqrt{2}\right) + \left(3\sqrt{2}\right)^2 = 12 - 12\sqrt{6} + 18 = 30 - 12\sqrt{6}$$

(ii) Express $\dfrac{20\sqrt{2}}{\sqrt{20}}$ in the form $a\sqrt{b}$, where a and b are integers and b is as small as possible.

Ans: $\dfrac{20\sqrt{2}}{\sqrt{20}} = \dfrac{20\sqrt{2}\sqrt{20}}{\sqrt{20}\sqrt{20}} = \dfrac{20\sqrt{40}}{20} = \sqrt{40} = \sqrt{4}\sqrt{10} = 2\sqrt{10}$

Ex-1-5: Make r the subject of $z = \pi r^2(x - y)$, where $r > 0$.

Ans: $z = \pi r^2(x - y)$

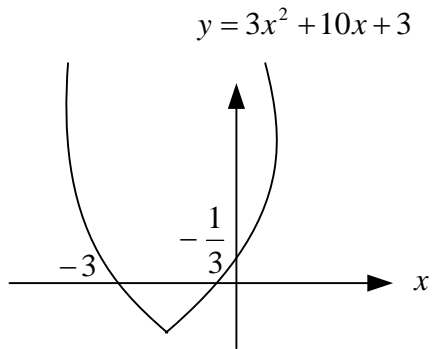$$\pi r^2 = \frac{z}{(x - y)}$$

$$r^2 = \frac{z}{\pi(x - y)}$$

$$r = \pm\sqrt{\frac{z}{\pi(x - y)}} \qquad\qquad \Rightarrow r = \sqrt{\frac{z}{\pi(x - y)}} \qquad \textbf{OK}$$

Ex-1-6: Solve the inequality $3x^2 + 10x + 3 > 0$

Ans:

$$3x^2 + 10x + 3 > 0$$

$$(3x+1)(x+3) > 0$$

$$y = 3x^2 + 10x + 3$$

(3x + 1) (x + 3) > 0   (x + 3)   (3x + 1)

x = - 3   **x = - 1/3**

	-		**-**	**0** **+**
	- **0**	**+**		**+**

(3x + 1) (x + 3)   **+**   **-**   **+**

Ans: **- 3 > x   or   x > - 1/3**

Ex-1-7:   Solve the inequality

(i) $\dfrac{4x-5}{7} > 2x+1$

Ans: $4x - 5 > 7(2x+1)$

$4x - 5 > 14x + 7$

$4x - 14x > 7 + 5$

$-10x > 12$

$x < -\dfrac{12}{10}$ $\Rightarrow x < -\dfrac{6}{5}$

(ii) $\dfrac{7}{x-3} > 2$

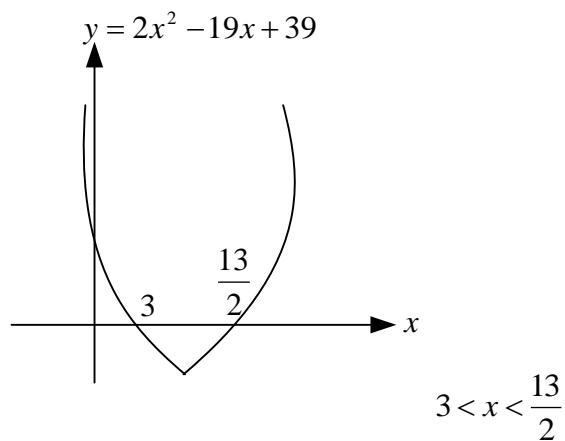**Ans:** $(x-3)^2 \dfrac{7}{x-3} > 2(x-3)^2$ **Very Important!**

$7(x-3) > 2(x-3)^2$

$7x-21 > 2(x^2-6x+9)$

$2(x^2-6x+9) < 7x-21$

$2x^2-12x+18 < 7x-21$

$2x^2-12x+18-7x+21 < 0$

$2x^2-19x+39 < 0$

$(2x-13)(x-3) < 0$

$y = 2x^2-19x+39$

$3 < x < \dfrac{13}{2}$

Ex-1-8:   Find the coefficient of x^5 in the binomial expansion of $(5+2x)^7$.

Note: Use the following, when **n** is **positive integer**!

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \binom{n}{4}a^{n-4}b^4 + \binom{n}{5}a^{n-5}b^5 + ... + \binom{n}{n}b^n$$

Note: Use the following, when **n** is **not integer**!

$$(1+x)^n = 1+nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 ...,$$

Hence, in this example, **n = 7**, then use the following:

$$\left(a+b\right)^{7} = \binom{7}{0}a^{7} + \binom{7}{1}a^{6}b + \binom{7}{2}a^{5}b^{2} + \binom{7}{3}a^{4}b^{3} + \binom{7}{4}a^{3}b^{4} + \binom{7}{5}a^{2}b^{5} + \binom{7}{6}ab^{6} + \binom{7}{7}b^{7}$$

Looking there properly: $\left(5+2x\right)^{7} = ... + \binom{7}{5}5^{2}\left(2x\right)^{5} + ...$

$$\binom{7}{5} = \frac{7!}{(7-5)!\,5!} = \frac{7\times6\times5!}{2!\times5!} = 21 \qquad \textbf{and} \qquad 5^{2}\left(2x\right)^{5} = 25\times2^{5}\,x^{5} = 800x^{5}$$

$$\Rightarrow \binom{7}{5}5^{2}\left(2x\right)^{5} = 21\times800x^{5} = 16800x^{5} \,...\textbf{ans: 16800}$$

Ex-1-9: You are given that $f(x) = 4x^{3} + kx + 6,$ where k is a constant. When f(x) is divided by $\left(x-2\right)$, the remainder is 42. Use the remainder theorem to find the value of k. Hence find a root of $f(x) = 0$.

Ans: $f(x) = 4x^{3} + kx + 6$

$f(2) = 4\times2^{3} + k\times2 + 6 = 42$

$4\times2^{3} + k\times2 + 6 = 42 \Rightarrow 32 + 2k + 6 = 42 \Rightarrow 2k = 42 - 38 = 4 \Rightarrow k = 2$

$f(x) = 4x^{3} + 2x + 6$

$f(-1) = 4(-1)^{3} + 2(-1) + 6 = -6 + 6 = 0 \Rightarrow x = -1$ **is a root**

Ex-1-10: You are given that k, k + 1 and k + 2 are three consecutive integers.

(i) Expand and simplify $k^{2} + \left(k+1\right)^{2} + \left(k+2\right)^{2}$

$k^{2} + \left(k+1\right)^{2} + \left(k+2\right)^{2} = k^{2} + k^{2} + 2k + 1 + k^{2} + 4k + 4 = 3k^{2} + 6k + 5$

(ii) For what values of n will the sum of the squares of these three consecutive integers be an even number?

$k^{2} + \left(k+1\right)^{2} + \left(k+2\right)^{2} = 3k^{2} + 6k + 5 = 3k\left(k+2\right) + 5$ **...(Eq.1)**
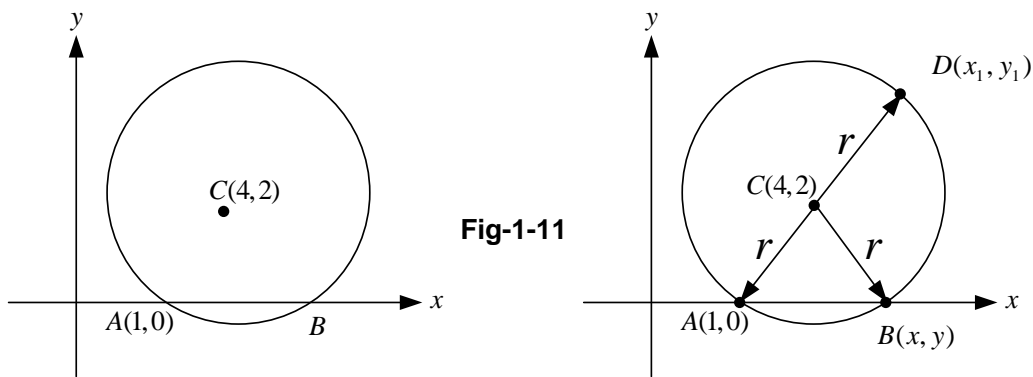
It is **even** for $k = 1, 3, 5, ...k = odd$

Give a reason for your answer.

Reason: $ans: \quad 3k\left(k+2\right) + 5$

$3k(k+2)$ is odd for n = odd, and $3k(k+2)$ is even for n = even, but 5 is added at the end. Then, $3k(k+2)+5$ is even for n = odd.

Ex-1-11: Fig-1-11 shows a sketch of a circle with centre C (4, 2). The circle intersects the *x*-axis at A(1, 0) and at B.

Fig-1-11

(i) Write down the coordinates of B.

Ans:

$$r = \sqrt{(4-x)^2 + (2-0)^2} = \sqrt{13}$$

$$r^2 = (4-x)^2 + (2-0)^2 = 13$$

$$(4-x)^2 = 13 - 4 = 9$$

$$(4-x) = \pm 3$$

(a) $\quad 4-x = 3 \Rightarrow -x = -1 \Rightarrow x = 1$

(b) $\quad 4-x = -3 \Rightarrow -x = -7 \Rightarrow x = 7$

$$\Rightarrow B(7,0)$$

(ii) Find the radius of the circle and hence write down the equation of the circle.

Ans: **Already found**

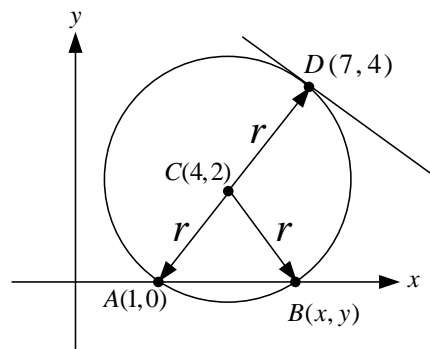**(iii)** AD is a diameter of the circle. Find the coordinates of D.

Ans: $4 = \dfrac{x_1 + 1}{2} \Rightarrow x_1 = 8 - 1 = 7$

$2 = \dfrac{y_1 + 0}{2} \Rightarrow y_1 = 4$

$\Rightarrow D(7,4)$

(iv)   Find the equation of the tangent to the circle at D. Give your answer in the form $y = ax + b$.

Ans:

Gradient of $CD = m = \dfrac{4 - 2}{7 - 4} = \dfrac{2}{3} \Rightarrow m_2 = -\dfrac{1}{m_1} = -\dfrac{3}{2}$

$CD = m = \dfrac{4 - 2}{7 - 4} = \dfrac{2}{3} \Rightarrow m_2 = -\dfrac{1}{m_1} = -\dfrac{3}{2}$

$4 = -\dfrac{3}{2}(7) + c \Rightarrow c = 4 + \dfrac{21}{2} = \dfrac{29}{2}$

$y = -\dfrac{3}{2}x + \dfrac{29}{2}$

Ex-1-12:   Fig-1-12 shows a sketch of the curve with equation $y = (x - 4)^2 - 3$
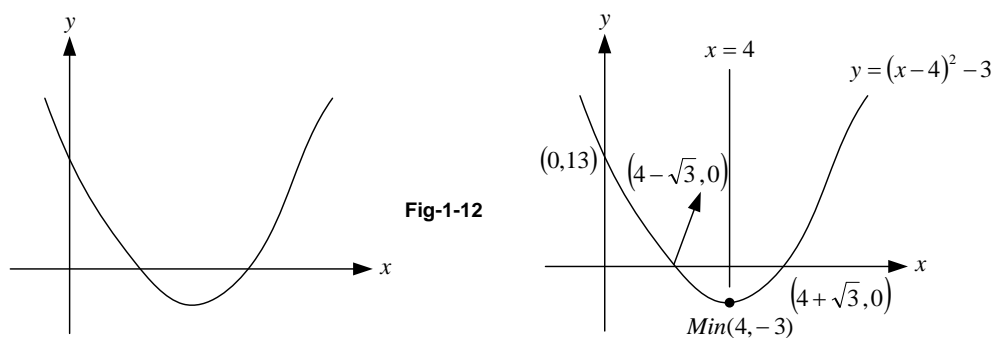
Fig-1-12

(i) Write down the equation of the line of symmetry of the curve and the coordinates of the minimum point. See Fig-1-12.

(ii) Find the coordinates of the points of intersection of the curve with the *x*-axis and the *y*-axis, using surds where necessary. See Fig-1-12.

(iii) The curve is translated by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Show that the equation of the translated

curve may be written as $y = x^2 - 12x + 33$

Ans: $y = f(x) = (x-4)^2 - 3 = x^2 - 8x + 16 - 3 = x^2 - 8x + 13$

$f(x-2) = (x-2)^2 - 8(x-2) + 13 = x^2 - 4x + 4 - 8x + 16 + 13$

$f(x-2) = x^2 - 12x + 33$

(iv) Show that the line $y = 8 - 2x$ meets the curve $y = x^2 - 12x + 33$ at just one point, and find the coordinates of this point.

Ans: $y = 8 - 2x$

$y = x^2 - 12x + 33$

$\Rightarrow x^2 - 12x + 33 = 8 - 2x \Rightarrow x^2 - 12x + 33 - 8 + 2x = 0 \Rightarrow x^2 - 10x + 25 = 0$

$\Rightarrow (x-5)^2 = 0 \Rightarrow x_1 = x_2 = 5 \Rightarrow P(5,-2)$ **only point.**

Ex-1-13: Fig-1-13 shows the graph of a cubic curve. It intersects the axes at (-5, 0),
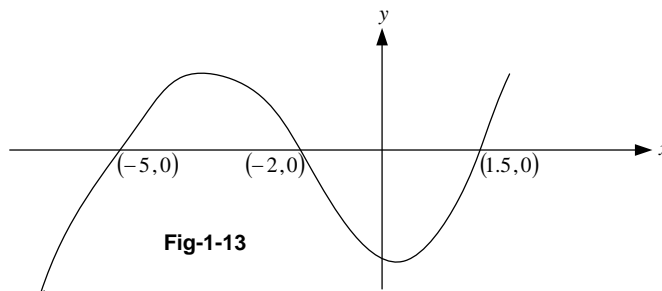
(-2, 0), (1.5, 0) and (0, -30).

Fig-1-13

(i) Use the intersections with both axes to express the equation of the curve in a factorised form.

Ans:

$(x+5)(x+2)(x-1.5)=0$

(ii)   Hence show that the equation of the curve may be written as

$$y = 2x^3 + 11x^2 - x - 30$$

(iii)   Draw the line $y = 5x + 10$ accurately on the graph. The curve and this line intersect at (-2, 0); find graphically the *x*-coordinates of the other points of intersection.

(iv)   Show algebraically that the *x*-coordinates of the other points of intersection satisfy the equation $2x^2 + 7x - 20 = 0$.

Hence find the exact values of the *x*-coordinates of the other points of intersection

PAPER-2

SECTION A(C1-12-1-05)

Ex-2-1:   Solve the inequality $2(x - 3) < 6x + 15$.

Ans:

$$2(x - 3) < 6x + 15.$$

$$2x - 6 < 6x + 15.$$

$$2x - 6x < 15 + 6.$$

$$-4x < 21.$$

$$-x < \frac{21}{4}$$

$$x > -\frac{21}{4}$$

Ex-2-2:   Make r the subject of $V = \dfrac{4}{3}\pi r^3$

Ans: $V = \dfrac{4}{3}\pi r^3$

$$r^3 = \frac{3V}{4\pi} \qquad \Rightarrow r = \sqrt[3]{\frac{3V}{4\pi}}$$

Ex-2-3: Find the coefficient of x^3 in the expansion of $(2+3x)^5$.

Ans: $(a+b)^n = \dfrac{a^n}{0!} + \dfrac{n \times a^{n-1}b}{1!} + \dfrac{n(n-1) \times a^{n-2}b^2}{2!} + \dfrac{n(n-1)(n-2) \times a^{n-3}b^3}{3!} + \ldots + b^n$

$(2+3x)^5 = \dfrac{2^5}{0!} + \dfrac{5 \times 2^4(3x)}{1!} + \dfrac{5(4) \times 2^3(3x)^2}{2!} + \dfrac{5(4)(3) \times 2^2(3x)^3}{3!} + \dfrac{5(4)(3)(2) \times 2(3x)^4}{4!} + (3x)^5$

$\Rightarrow (2+3x)^5 = 2^5 + 5(2)^4(3x) + 10(2)^3(3x)^2 + 10(2)^2(3x)^3 + 5(2)(3x)^4 + (3x)^5$

The coefficient of x^3 is: $\quad 1080 \; x^3$

OR the following method is very good, always use this , if n is positive integer!!

$(a+b)^5 = \binom{5}{0}a^5 + \binom{5}{1}a^4b + \binom{5}{2}a^3b^2 + \binom{5}{3}a^2b^3 + \ldots$

Looking there properly: $\qquad (2+3x)^5 = \ldots + \binom{5}{3}2^2(3x)^3 + \ldots$

$\binom{5}{3} = \dfrac{5!}{(5-3)!\,3!} = \dfrac{5 \times 4 \times 3!}{2! \times 3!} = \dfrac{20}{2} = 10$ **and** $2^2(2x)^3 = 4 \times 3^3 x^3 = 4 \times 27 x^3 = 108 x^3$

$\Rightarrow \binom{5}{3}2^2(3x)^3 = 10 \times 4 \times 27 x^3 = 1080 x^3 \ldots$ **ans: 1080**

Ex-2-4: Find the value of the following.

(i) $\left(\dfrac{1}{2}\right)^{-2} = \dfrac{1}{\left(\dfrac{1}{2}\right)^2} = \dfrac{2^2}{1} = 4$

(ii) $32^{\frac{4}{5}} = (2^5)^{\frac{4}{5}} = 2^4 = 16$

Ex-2-5: The line L is parallel to $y = -3x + 5$ and passes through the point (4, 3)

Find the coordinates of the points of intersection of L with the axes.

Ans: The gradient of L is: $m = -3$, the equation is: $y = -3x + b$

$3 = -3(4) + b \implies b = 3 + 12 = 15 \qquad \implies y = -3x + 15$

The line L crosses the x-axis when $y = 0$, $-3x + 15 = 0 \implies 3x = 15 \implies x = 5$

Point is: **(5,0)**

The line L crosses the y-axis when $x = 0$, $y = 15$

Point is: **(0,15)**

Ex-2-6:   Express $x^2 - 6x$ in the form $(x - a)^2 - b$

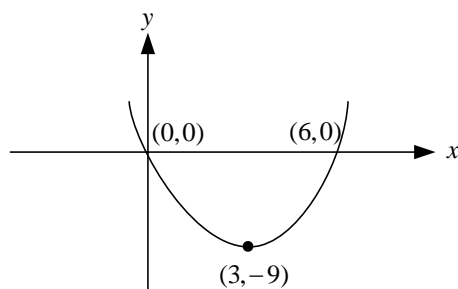Sketch the graph of $y = x^2 - 6x$, giving the coordinates of its minimum point and the intersection with the axes.

Ans: $x^2 - 6x + 9 - 9 = (x - 3)^2 - 9$

The coordinates of its **minimum** point (3,-9)

The intersection with the axes. $y = (x - 3)^2 - 9$

The intersection with the x-axis. $y = (x - 3)^2 - 9 = 0 \implies x = 0, \quad x = 6$

The intersection with the y-axis. $\implies x = 0, \quad y = 0$

Ex-2-7:   Find, in the form $y = mx + c$, the equation of the line passing through A(3, 7) and B(5,-1).

Ans: $y = mx + c$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 7}{5 - 3} = \frac{-8}{2} = -4$$

$$y = -4x + c$$

$$7 = -4 \times 3 + c$$

$$c = 7 + 12 = 19$$

$$\Rightarrow y = -4x + 19$$

Show that the midpoint of AB lies on the line $x + 2y = 10$.

Midpoint of AB = $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2} \right) = \left(\dfrac{3+5}{2}, \dfrac{7-1}{2} \right) = (4, 3)$

$$x + 2y = 10$$

$$4 + 2 \times 3 \overset{?}{=} 10 \Rightarrow 10 = 10 \Rightarrow OK$$

Ex-2-8: Simplify $(2 + \sqrt{3})(2 - \sqrt{3})$

Ans: $(2 + \sqrt{3})(2 - \sqrt{3}) = 2^2 - \left(\sqrt{3} \right)^2 = 4 - 3 = 1$

Express $\dfrac{1 + \sqrt{2}}{3 - \sqrt{2}}$ in the form $a + b\sqrt{2}$, where a and b are rational.

$$\frac{1 + \sqrt{2}}{3 - \sqrt{2}} = \frac{\left(1 + \sqrt{2}\right)\left(3 + \sqrt{2}\right)}{\left(3 - \sqrt{2}\right)\left(3 + \sqrt{2}\right)} = \frac{3 + \sqrt{2} + 3\sqrt{2} + \left(\sqrt{2}\right)^2}{3^2 - \left(\sqrt{2}\right)^2} = \frac{5 + 4\sqrt{2}}{7} = \frac{5}{7} + \frac{4}{7}\sqrt{2}$$

SECTION B (P-2)

Ex-2-9: Fig-2-9 shows a circle with centre C(2,1) and radius 5.
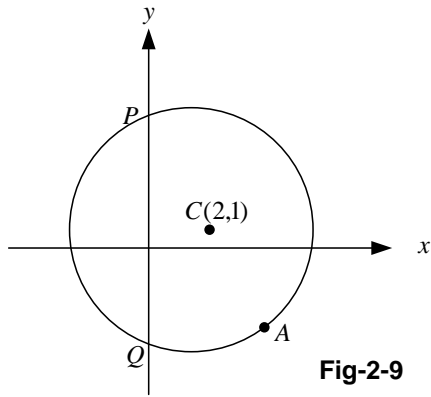
Fig-2-9

(i) Show that the equation of the circle may be written as

$$x^2 + y^2 - 4x - 2y - 20 = 0.$$
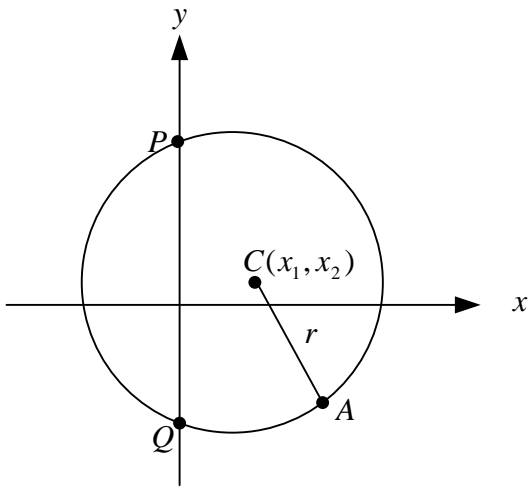
$$\left(x - x_1\right)^2 + \left(y - y_1\right)^2 = r^2$$

$$\left(x - 2\right)^2 + \left(y - 1\right)^2 = 5^2$$

$$x^2 - 4x + 4 + y^2 - 2y + 1 = 25$$

$$x^2 + y^2 - 4x - 2y = 25 - 5 = 20$$

$$x^2 + y^2 - 4x - 2y = 20$$

$$x^2 + y^2 - 4x - 2y - 20 = 0$$

(ii) Find the coordinates of the points P and Q where the circle cuts the y-axis. Leave your answers in the form $a \pm \sqrt{b}$

$$x^2 + y^2 - 4x - 2y - 20 = 0$$

$$0^2 + y^2 - 4 \times 0 - 2y = 20 \Rightarrow y^2 - 2y - 20 = 0$$

$$y_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times (-20)}}{2 \times 1} = \frac{2 \pm \sqrt{4 - 80}}{2} = \frac{2 \pm \sqrt{76}}{2} = \frac{2 \pm 6\sqrt{2}}{2} = 1 + 3\sqrt{2}$$

(iii)   Verify that the point A(5,-3) lies on the circle.

$$x^2 + y^2 - 4x - 2y - 20 = 0$$

$$5^2 + (-3)^2 - 4(5) - 2(-3) - 20 \overset{?}{=} 0$$

$$25 + 9 - 20 + 6 - 20 \overset{?}{=} 0 \Rightarrow OK$$

Show that the tangent to the circle at A has equation $4y = 3x - 27$
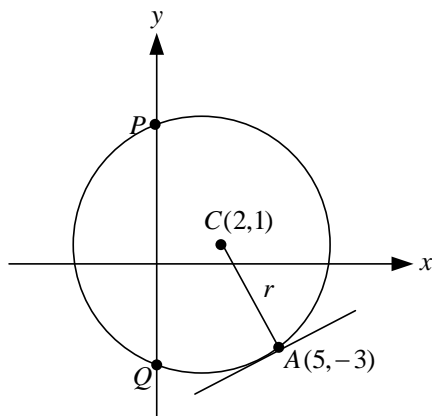
$$y = mx + b$$

Gradient of $AC = m = \dfrac{1-(-3)}{2-5} = \dfrac{4}{-3} = -\dfrac{4}{3}$

Gradient of $= m_1 = -\dfrac{1}{m} = \dfrac{3}{4}$

Equation of tangent: $\quad y = mx + b = \dfrac{3}{4}x + b$

$$-3 = \frac{3}{4}(5) + b \Rightarrow b = -3 - \frac{15}{4} = -\frac{27}{4}$$

$$y = \frac{3}{4}x - \frac{27}{4}$$

$$\Rightarrow 4y = 3x - 27$$

Ex-2-10: A cubic polynomial is given by $f(x) = x^3 + x^2 - 10x + 8$

    (i)   Show that (x-1) is a factor of f(x).
           Factorise f(x) fully

           Sketch the graph of y=f(x).

Ans: if (x-1) is a factor of f(x), then f(1)=0

$$f(x) = x^3 + x^2 - 10x + 8 \Rightarrow f(1) = 1^3 + 1^2 - 10(1) + 8 = 1 + 1 - 10 + 8 = 0 \Rightarrow OK$$

$$
\begin{array}{r}
x^2 + 2x - 8 \\
\hline
x-1 \,\big|\; x^3 + x^2 - 10x + 8 \\
x^3 - x^2 \\
\underline{\;\; -\;\; +\;\;} \\
2x^2 - 10x \\
2x^2 - 2x \\
\underline{\;\; -\;\; +\;\;} \\
-8x + 8 \\
-8x + 8 \\
\underline{\;\; +\;\; -\;\;} \\
0
\end{array}
$$

$$\Rightarrow f(x) = x^3 + x^2 - 10x + 8 = (x-1)(x^2 + 2x - 8) = (x-1)(x+4)(x-2)$$
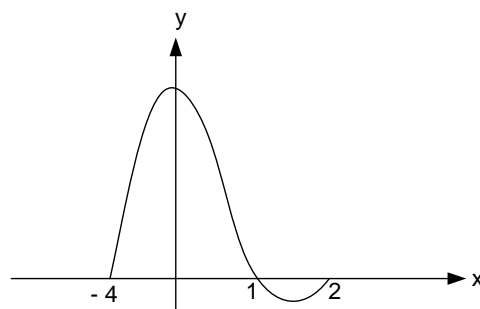
    (ii)   The graph of y=f(x) is translated by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$.

Write down an equation for the resulting graph. You need not simplify your answer.
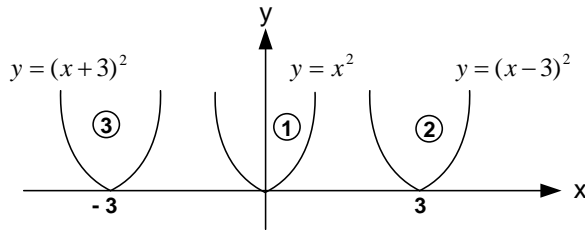
$$y = f(x) = x^2 \qquad \text{①}$$

The graph of y=f(x) is translated by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ $\qquad y = f(x-3) = (x-3)^2 \qquad \text{②}$

The graph of y=f(x) is translated by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ $\qquad y = f(x+3) = (x+3)^2 \qquad \text{③}$

Note: The graph of y=f(x) is translated by $\begin{pmatrix} a \\ b \end{pmatrix}$ $\quad y - b = f(x-a) = (x-a)^2$

$$\Rightarrow f(x) = x^3 + x^2 - 10x + 8 = (x-1)(x^2 + 2x - 8) = (x-1)(x+4)(x-2)$$

$$\Rightarrow f(x+3) = (x+3)^3 + (x+2)^2 - 10(x+3) + 8$$

Extra Information

$$y = f(x) = x \qquad\qquad\qquad y = f(x) = x \qquad \text{①}$$

The graph of y=f(x) is translated by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ $\qquad y = f(x-3) \qquad \text{②}$
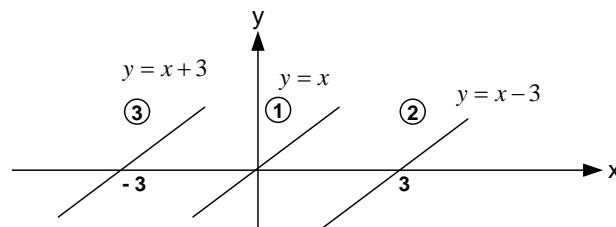
The graph of y=f(x) is translated by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ $\qquad y = f(x-(-3)) = f(x+3) \quad \text{③}$

Note: The graph of y=f(x) is translated by $\begin{pmatrix} a \\ 0 \end{pmatrix}$ $\qquad y = f(x-a)$

Note: The graph of y=f(x) is translated by $\begin{pmatrix} a \\ b \end{pmatrix}$ $\qquad y - b = f(x-a)$

Find also the intercept on the y-axis of the resulting graph.

$$\Rightarrow f(x) = (x+3)^3 + (x+2)^2 - 10(x+3) + 8$$

$$\Rightarrow f(0)=(0+3)^3+(0+2)^2-10(0+3)+8=9+4-30+8=-9$$

Intercept (0, -9)

Ex-2-11: (i) Show that the graph of $y=x^2-5x+21$ is above the x-axis for all values of x.

Ans: $y=x^2-5x+21=x^2-5x+\dfrac{25}{4}+21-\dfrac{25}{4}=\left(x-\dfrac{5}{2}\right)^2+\dfrac{59}{4}$

The minimum is $\dfrac{59}{4}$ when x = 0, hence always above x-axis

(iii) Find the set of values of x for which the graph of $y=2x^2+x-10$ is above the x-axis.

$$2x^2+x-10>0 \qquad \Rightarrow (2x+5)(x-2)>0$$

(2x + 5) (x - 2) > 0	(2x + 5)		(x - 2)
	$x=-\dfrac{5}{2}$		**x = 2**
(x - 2)	**-**	**-**	**0** **+**
(2x + 5)	**-** **0**	**+**	**+**
(2x + 5)(x - 2)	**+**	**-**	**+**

Ans: $x<-\dfrac{5}{2}$ **OR** **x > 2**

(iv) Find algebraically the coordinates of the points of intersection of the graph of $y=x^2-3x+11$ and $y=2x^2-x-10$

Ans: $y=2x^2+x-10$ and $y=x^2-3x+11$

$$x^2-3x+11=2x^2+x-10$$

$$x^2-2x^2-3x-x+11+10=0$$

$$-x^2-4x+21=0 \Rightarrow x^2+4x-21=0 \Rightarrow (x-7)(x+3)=0$$

$$\Rightarrow x=7, \; and \; x=-3 \qquad \text{**Points are:**} \; (7,95) \; \text{and} \; (-3,5)$$

PAPER-3

Ex-3-1: Find the remainder when $x^3 + 2x^2 - 5$ is divided by x-3.

Ans: $f(x) = x^3 + 2x^2 - 5$

$$\Rightarrow f(3) = (3)^3 + 2 \times (3)^2 - 5 = 27 + 18 - 5 = 45 - 5 = 40$$

Hence the remainder is 40 as shown below for verification

$$
\begin{array}{r}
x^2+5x+15 \\
\hline
\text{X-3} \,\big|\, X^3+2x^2-5 \\
-\;X^3-3x^2 \\
+ \\
\hline
5x^2-5 \\
5x^2-15x \\
-\quad+ \\
\hline
15x-5 \\
15x-45 \\
-\quad+ \\
\hline
40
\end{array}
$$

Ex-3-2: Make x the subject of $4x - 7y = 2y - mx$

Ans: $4x - 7y = 2y - mx \qquad \Rightarrow 4x + mx = 2y + 5y = 7y$

$$\Rightarrow (4+m)x = 7y \qquad \Rightarrow x = \frac{7y}{4+m}$$

Ex-3-3: The smallest of three consecutive integer is n.

Write down the other two integers.

Prove that the sum of any three consecutive integers is divisible by 3.

Ans: If n is the smallest of the 3 consecutive numbers:

1st number =n

2nd number=n+1

3rd number=n+2

Sum of the 3 consecutive numbers is: n+(n+1)+(n+2)=3n+3. If divided by 3, it is equal to n+1.

Ex-3-4: A line has equation $5x + 7y = 15$. Find its gradient and the coordinates of the points where it crosses the axes.

Ans: The standard equation of a straight line is: $y = mx + b$, where m is the **gradient** and b is the **y-intercept**.

$$5x + 7y = 15 \qquad \Rightarrow 7y = -5x + 15 \qquad \Rightarrow y = -\frac{5}{7}x + \frac{15}{7}$$

Hence, $m = -\dfrac{5}{7}$, and the point where it crosses the y-axis, i.e. x=0

$y = \dfrac{15}{7}$, and the point is $\left(0, \dfrac{15}{7}\right)$. The point where it crosses the x-axis, i.e.

y=0, $\Rightarrow y = -\dfrac{5}{7}x + \dfrac{15}{7} = 0 \qquad \Rightarrow 5x = 15 \Rightarrow x = 3$, and the point is $(3, 0)$.

Ex-3-5: Find the binomial expansion of $(2-x)^3$.

Ans: $(a-b)^n = \dfrac{a^n}{0!} - \dfrac{n \times a^{n-1}b}{1!} + \dfrac{n(n-1) \times a^{n-2}b^2}{2!} - \dfrac{n(n-1)(n-2) \times a^{n-3}b^3}{3!} + \dots - b^n$

$$\Rightarrow (2-x)^3 = \frac{2^3}{0!} - \frac{3 \times 2^2 x}{1!} + \frac{2 \times 3 \times 2x^2}{2!} - \frac{1 \times 2 \times 3 \times 2^0 x^3}{3!} = 8 - 12x + 6x^2 - x^3$$

OR the following method is very good, always use this instead, if n positive integer!!

$$(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$$

$$(2-x)^3 = \binom{3}{0}2^3 - \binom{3}{1}2^2 x + \binom{3}{2}2x^2 - \binom{3}{3}x^3$$

$$\binom{3}{0} = \frac{3!}{(3-0)!\,0!} = \frac{3!}{3! \times 0!} = 1 \qquad\qquad \binom{3}{1} = \frac{3!}{(3-1)!\,1!} = \frac{3!}{2! \times 1!} = 3$$

$$\binom{3}{2} = \frac{3!}{(3-2)!\,2!} = \frac{3!}{1! \times 2!} = 3 \qquad\qquad \binom{3}{3} = \frac{3!}{(3-3)!\,3!} = \frac{3!}{0! \times 3!} = 1$$

$$(2-x)^3 = \binom{3}{0}2^3 - \binom{3}{1}2^2 x + \binom{3}{2}2x^2 - \binom{3}{3}x^3$$

$$(2-x)^3 = 2^3 - 3 \times 2^2 x + 3 \times 2x^2 - x^3$$

$$(2-x)^3 = 8 - 12x + 6x^2 - x^3$$

Ex-3-6:   Simplify the following.

(i) b^0

Ans: $b^0 = 1$

(ii) $a^7 \div a^{-3} = a^{7-(-3)} = a^{7+3} = a^{10}$

(iii) $\left(9a^6b^2\right)^{-\frac{1}{2}} = \dfrac{1}{\left(9a^6b^2\right)^{\frac{1}{2}}} = \dfrac{1}{\sqrt{9a^6b^2}} = \dfrac{1}{3a^3b}$

Ex-3-7:   (i)   Simplify $\sqrt{24} + \sqrt{3}$

Ans: $\sqrt{27} + \sqrt{3} = \sqrt{9 \times 3} + \sqrt{3} = \sqrt{9} \times \sqrt{3} + \sqrt{3} = 3\sqrt{3} + \sqrt{3} = 4\sqrt{3}$

(ii)   Express $\dfrac{36}{5-\sqrt{7}}$ in the form $a + b\sqrt{7}$, where a and b are integers

Ans: $\dfrac{36}{5-\sqrt{7}} = \dfrac{36\left(5+\sqrt{7}\right)}{\left(5-\sqrt{7}\right)\left(5+\sqrt{7}\right)} = \dfrac{36\left(5+\sqrt{7}\right)}{5^2 - \left(\sqrt{7}\right)^2} = \dfrac{36\left(5+\sqrt{7}\right)}{25-7} = \dfrac{36\left(5+\sqrt{7}\right)}{18} = 2\left(5+\sqrt{7}\right)$
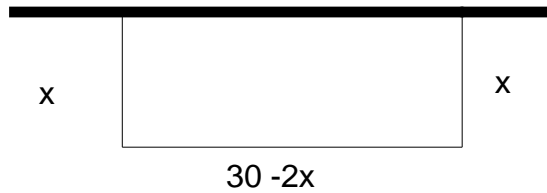
$$= 10 + 2\sqrt{7}$$

Ex-3-8:   **Fig.8** is a plan view of a rectangular enclosure. A wall forms one side of the enclosure. The other three sides are formed by fencing of total length 30m.

The width of the rectangle is x m and the area enclosed is 112 m^2 .

Fig-3-8

$$x \qquad \qquad \qquad x$$

$$30-2x$$

Area=

$$x(30-2x)=112 \Rightarrow -2x^2+30x=112 \Rightarrow -2x^2+30x-112=0 \Rightarrow x^2-15x+56=0$$

Show that $x^2-15x+56=0$.

By factorising, solve this equation and find the possible dimensions of the rectangle.

$$\Rightarrow x^2-15x+56=(x-8)(x-7)=0 \Rightarrow x=8 \quad and \quad x=7 \qquad \text{7 by 16 or 8 by 14}$$

Ex-3-9: Find the x-coordinates of the points of intersection of the line $y=3x+2$ and the curve $y=3x^2-7x+1$. Leave your answers in surd form.

Ans: At the point of intersection, the values of x and y of both the straight line and the curve are the same.

$$\Rightarrow y=3x+2=3x^2-7x+1 \qquad \Rightarrow 3x^2-7x+1-3x-2=0$$
$$\Rightarrow 3x^2-10x-1=0$$

$$\Rightarrow x_{1,2}=\frac{-b\pm\sqrt{b^2}}{2a}=\frac{-(-10)\pm\sqrt{(-10)^2-4(3)(-1)}}{2(3)}=\frac{10\pm\sqrt{100+12}}{6}$$

$$\Rightarrow \frac{10\pm\sqrt{112}}{6}=\frac{10\pm\sqrt{16\times7}}{6}=\frac{10\pm\sqrt{16}\sqrt{7}}{6}=\frac{10\pm4\sqrt{7}}{6}=\frac{5\pm2\sqrt{7}}{3}$$
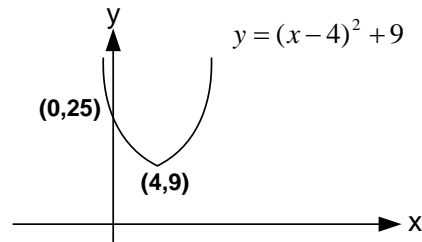
SECTION B (P-3)

Ex-3-10: (i) Write $x^2-8x+25$ in the form $(x-a)^2+b$

Ans: $x^2-8x+25=x^2-8x+16-16+25=(x-4)^2+9$

(ii) State the coordinates of the minimum point on the graph of $y=x^2-8x+25$ and sketch this graph.

$$y=x^2-8x+25=x^2-8x+16-16+25=(x-4)^2+9$$

The minimum value of $y = 9$ when $x = 4$. Hence the coordinates of the minimum point is: (4,9)

(iii)   Solve the inequality $x^2 - 9x + 25 > 11$

$x^2 - 9x + 25 > 11 \qquad \Rightarrow x^2 - 9x + 25 - 11 > 0$

$\Rightarrow x^2 - 9x + 14 > 0 \qquad \Rightarrow (x-7)(x-2) > 0$

(x – 7) (x - 1) > 0   (x - 1)   (x - 7)

X = 1   **X = 7**

(x - 7)	**-**		**-**	**0**	**+**
(x - 1)	**-**	**0**	**+**		**+**
(x – 7)(x - 1)	**+**		**-**		**+**

Ans: x < 1 OR x > 7

(iv)   The graph of $y = x^2 - 8x + 25$ is translated by $\begin{pmatrix} 0 \\ -20 \end{pmatrix}$. State an equation for the resulting graph.

Ans: replace **y** by **y + 20** i.e.

$y - (-20) = x^2 - 8x + 25 \Rightarrow y + 20 = x^2 - 8x + 25 \Rightarrow y = x^2 - 8x + 25 - 20$
$\Rightarrow y = x^2 - 8x + 5$

Note: Consider the followings:

$y = f(x)$ is **translated** by $\begin{pmatrix} a \\ b \end{pmatrix}$

Replace $x \to x - a$ **and replace** $y \to y - b$

Ex-3-11: The graph of $y = x^2 - 8x + 25$ is translated by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

$$y - (-2) = (x-3)^2 - 8(x-3) + 25$$

$$y + 2 = x^2 - 6x + 9 - 8x + 24 + 25$$

$$y = x^2 - 14x + 9 + 24 + 25 - 2$$

$$y = x^2 - 14x + 56$$

Ex-3-12: The graph of $y = x^2 - 8x + 25$ is translated by $\begin{pmatrix} -3 \\ -2 \end{pmatrix}$.

Replace $x \rightarrow x - (-3) \Rightarrow x \rightarrow x + 3$

And replace $y \rightarrow y - (-2) \Rightarrow y \rightarrow y + 2$

$$y - (-2) = (x+3)^2 - 8(x+3) + 25$$

$$y + 2 = x^2 + 6x + 9 - 8x - 24 + 25$$

$$y = x^2 + 6x + 9 - 8x - 24 + 25 - 2$$

$$y = x^2 - 2x + 8$$

Ex-3-13: The points A(0,2), B(7,9) and C(6,10) lie on the circumference of a circle as shown in Fig-3-13

Fig-3-13
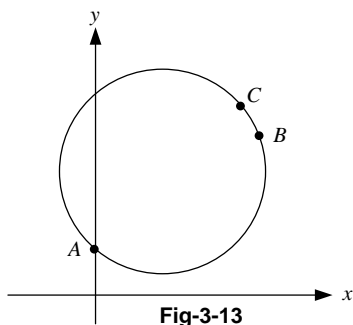
(i)   Find the length of AC.
       Prove that triangle ABC is right-angled at B.

$$AC = \sqrt{(6-0)^2 + (10-2)^2} = \sqrt{36+64} = \sqrt{100} = 10$$

$$AB = \sqrt{(7-0)^2 + (9-2)^2} = \sqrt{49+49} = \sqrt{98}$$

$$BC = \sqrt{(6-7)^2 + (10-9)^2} = \sqrt{1+1} = \sqrt{2}$$

$$AC^2 = AB^2 + BC^2 = 100 = 98 + 2$$

(ii)   Hence show that the centre of the circle is (3,6) and its radius is 5. Find the equation of the circle.

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

AC is the diameter

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left(\frac{0+6}{2}, \frac{2+10}{2} \right) = (3,6)$$

$$r = \frac{AC}{2} = \frac{10}{2} = 5$$

$$(x-3)^2 + (y-6)^2 = 5^2$$

(iii)   Find an equation for the tangent to the circle at C.

$$y = mx + b$$

Gradient of $AC = m = \dfrac{10-2}{6-0} = \dfrac{8}{6} = \dfrac{4}{3}$

Gradient of $= m_1 = -\dfrac{1}{m} = -\dfrac{4}{3}$

Equation oftangent: $\quad y = mx + b = -\dfrac{4}{3}x + b$

$10 = -\dfrac{4}{3}(6) + b \Rightarrow b = 10 + 8 = 18$
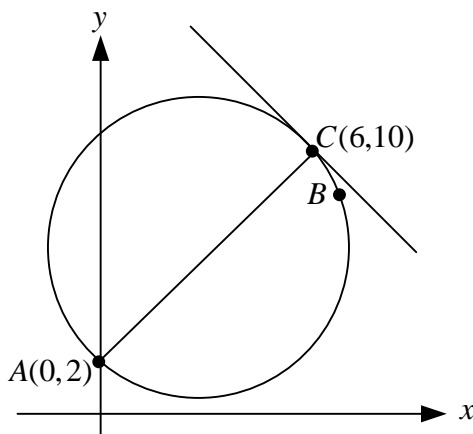
$y = -\dfrac{4}{3}x + 18$

Find the coordinates of the points where this tangent crosses the axes.

Ans: When this tangent crosses the x-axis, then y = 0

$y = -\dfrac{4}{3}x + 18 = 0 \Rightarrow \dfrac{4}{3}x = 18 \Rightarrow 4x = 54 \Rightarrow x = \dfrac{54}{4} \Rightarrow P\left(\dfrac{54}{4}, 0\right)$

When this tangent crosses the y-axis, then x = 0

$y = -\dfrac{4}{3}x + 18 = -\dfrac{4}{3}(0) + 18 = 18 \Rightarrow P(0, 18)$

Ex-3-14: In the cubic polynomial f(x), the coefficient of x^3 is 1. The roots of f(x)=0, are -1, 2, and 5.

(i) Write f(x) in factorised form.

Ans: $f(x) = (x+1)(x-2)(x-5) = 0$

Show that f(x) may be written as $f(x) = x^3 - 6x^2 + 3x + 10$

Ans: $f(x) = (x+1)(x-2)(x-5) = 0 \Rightarrow$ Multiply the factors correctly, then you will

get $f(x) = x^3 - 6x^2 + 3x + 10$

(ii) Sketch the graph of y=f(x).
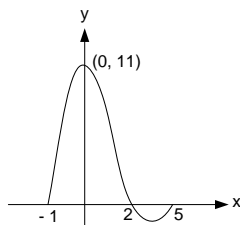
(iii) Show that x = 4 is one root of the equation f(x) + 10 = 0

Hence find a quadratic equation which is satisfied by the other two roots of the equation f(x) + 10 = 0

Ans: $f(x) = x^3 - 6x^2 + 3x + 10$

$f(x) + 10 = x^3 - 6x^2 + 3x + 10 + 10 = x^3 - 6x^2 + 3x + 20$

$\Rightarrow f(x) = x^3 - 6x^2 + 3x + 20$

$f(4) = 4^3 - 6 \times 4^2 + 3 \times 4 + 20 = 64 - 96 + 12 + 20 = 0$

$$
\begin{array}{r}
x^2 - 2x - 5 \\
\hline
x - 4 \,\big|\; x^3 - 6x^2 + 3x + 20 \\
\underline{x^3 - 4x^2} \\
-2x^2 + 3x \\
\underline{-2x^2 + 8x} \\
-5x + 20 \\
\underline{-5x + 20} \\
0
\end{array}
$$

$f(x) = x^3 - 6x^2 + 3x + 20 = (x - 4)(x^2 - 2x - 5)$

Hence, the quadratic equation is $x^2 - 2x - 5$

PAPER-4

SECTION A (C1-16-1-06)

Ex-4-1: n is a positive integer. Show that $n^2 + n$ is always even.

Ans: $n^2 + n = n(n+1)$

If n is even, n+1 is odd, then, *even* × *odd* = *even*

If n is odd, n+1 is even, then, *odd* × *even* = *even*

Ex-4-2: Fig-4-2 shows graphs A and B

Fig-4-2
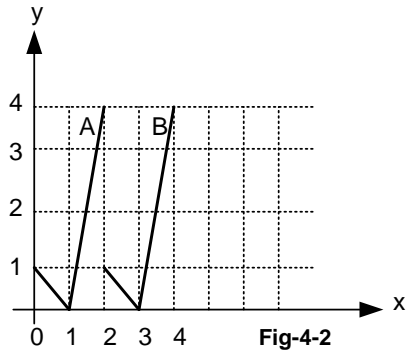
(i)   State the transformation which maps graph A onto graph B.

Ans: Graph A is translated by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

(ii)   The equation of graph A is y=f(x).
       Which one of the following is the equation of graph B?

y = f(x) + 2   y = f(x) – 2   y = f(x+2)   **y = f(x-2)**

y = 2f(x)   y = f(x+3)   y = f(x-3)   y = 3f(x)

Note: **Ans:**   **y = f(x-2) and see the following for more detail.**
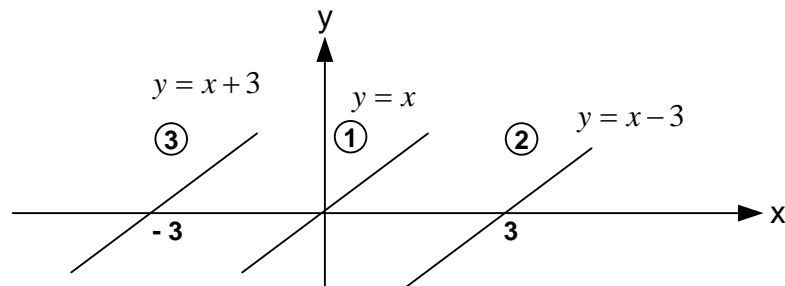
$y = f(x) = x$ $y = f(x) = x$ ①

The graph of y=f(x) is translated by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$ $y = f(x-3)$ ②

The graph of y=f(x) is translated by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ $y = f(x-(-3)) = f(x+3)$ ③

Note: The graph of y=f(x) is translated by $\begin{pmatrix} a \\ 0 \end{pmatrix}$ $y = f(x-a)$

Note: The graph of y=f(x) is translated by $\begin{pmatrix} a \\ b \end{pmatrix}$ $y - b = f(x-a)$

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Ex-4-3: Find the binomial expansion of $(2+x)^4$, writing each term as simply as possible.

Ans: $(a+b)^n = \dfrac{a^n}{0!} + \dfrac{n \times a^{n-1}b}{1!} + \dfrac{n(n-1) \times a^{n-2}b^2}{2!} + \dfrac{n(n-1)(n-2) \times a^{n-3}b^3}{3!} + \ldots + b^n$

$$(2+x)^4 = \frac{2^4}{0!} + \frac{4 \times 2^3 x}{1!} + \frac{4(3) \times 2^2 x^2}{2!} + \frac{4(3)(2) \times 2^1 x^3}{3!} + \frac{4(3)(2)x^4}{4!}$$

$$(2+x)^4 = 16 + 32x + 24x^2 + 8x^3 + x^4$$

OR the following method is very good, always use this when n is positive integer!!

$$(a+b)^4 = \binom{4}{0}a^4 + \binom{4}{1}a^3 b + \binom{4}{2}a^2 b^2 + \binom{4}{3}ab^3 + \binom{4}{4}b^4$$

$$(2+x)4 = \binom{4}{0}2^4 + \binom{4}{1}2^3 x + \binom{4}{2}2^2 x^2 + \binom{4}{3}2x^3 + \binom{4}{4}x^4$$

$$\binom{4}{0} = \frac{4!}{(4-0)!\,0!} = \frac{4!}{4! \times 0!} = 1 \qquad \binom{4}{1} = \frac{4!}{(4-1)!\,1!} = \frac{4!}{3! \times 1!} = 4$$

$$\binom{4}{2} = \frac{4!}{(4-2)!\,2!} = \frac{4!}{2! \times 2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = \frac{24}{4} = 6$$

$$\binom{4}{3} = \frac{4!}{(4-3)!\,3!} = \frac{4!}{01! \times 3!} = 4 \qquad \binom{4}{4} = \frac{4!}{(4-4)!\,4!} = \frac{4!}{01! \times 4!} = 1$$

$$(2+x)4 = \binom{4}{0}2^4 + \binom{4}{1}2^3 x + \binom{4}{2}2^2 x^2 + \binom{4}{3}2x^3 + \binom{4}{4}x^4$$

$$(2+x)4 = 2^4 + 4 \times 2^3 x + 6 \times 2^2 x^2 + 4 \times 2x^3 + x^4$$

$$(2+x)4 = 16 + 32x + 24x^2 + 8x^3 + x^4$$

Ex-4-4: Solve the inequality $\dfrac{3(2x+1)}{4} > -6$

Ans: $\dfrac{3(2x+1)}{4} > -6 \qquad \Rightarrow 3(2x+1) > -24 \qquad \Rightarrow 6x+3 > -24 \quad \Rightarrow 6x > -24-3$

$$\Rightarrow 6x > -27 \qquad \Rightarrow x > -\frac{27}{6} = -\frac{9}{2}$$

Ex-4-5: Make C the subject of the formula $P = \dfrac{C}{C+4}$

Ans: $P = \dfrac{C}{C+4} \Rightarrow PC + 4P = C \Rightarrow PC - C = -4P \Rightarrow C(P-1) = -4P$

$$\Rightarrow C = -\dfrac{4P}{P-1}$$

Ex-4-6: When $x^3 + 3x + k$ is divided by $x - 1$, the remainder is 6. Find the value of k.

Ans: $f(x) = x^3 + 3x + k \quad \Rightarrow f(1) = 1 + 3 + k = 6 \Rightarrow k = 6 - 4 = 2$

Ex-4-7: The line AB has equation $y = 4x - 5$ and passes through the point B(2, 3), as shown in Fig-4-7. The line BC is perpendicular to AB and cuts the x-axis at C. Find the equation of the line BC and the x-coordinate of C.

Fig-4-7
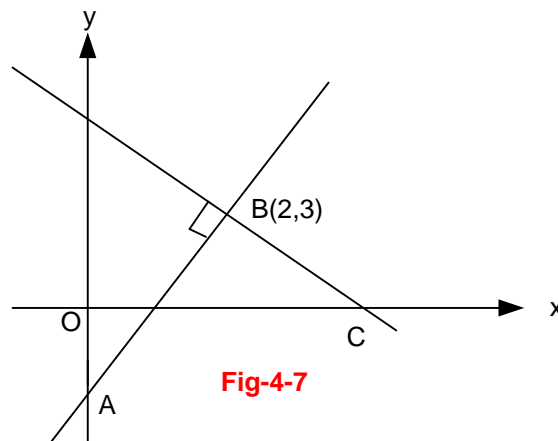
Ans: **The equation of line AB:** $y = 4x - 5$, the gradient is: $m = 4$. Because BC is perpendicular on AB, therefore, its $m = -\dfrac{1}{4}$. **The equation of line BC:**

$y = -\dfrac{1}{4}x + b$ and because it passes through B(2,3),

$3 = -\dfrac{1}{4}(2) + b \quad \Rightarrow b = 3 + \dfrac{2}{4} = 3 + \dfrac{1}{2} = \dfrac{7}{2} \quad \Rightarrow y = -\dfrac{1}{4}x + \dfrac{7}{2}$ is the equation of BC

Ex-4-8: (i) Simplify $5\sqrt{8} + 4\sqrt{50}$). Express your answer in the form $a\sqrt{b}$,

where a and b are integers and b is as small as possible.

Ans: $5\sqrt{8}+4\sqrt{50})=5\sqrt{4\times2}+4\sqrt{25\times2}=5\sqrt{4}\sqrt{2}+4\sqrt{25}\sqrt{2}=5\times2\sqrt{2}+4\times5\sqrt{2}$

$=10\sqrt{2}+20\sqrt{2}=30\sqrt{2}$

(ii)   Express $\dfrac{\sqrt{3}}{6-\sqrt{3}}$ in the form $p+q\sqrt{3}$, where p and q are rational.

Ans: $\dfrac{\sqrt{3}}{6-\sqrt{3}}=\dfrac{\sqrt{3}\left(6-\sqrt{3}\right)}{\left(6-\sqrt{3}\right)\left(6+\sqrt{3}\right)}=\dfrac{6\sqrt{3}-\left(\sqrt{3}\right)^2}{6^2-\left(\sqrt{3}\right)^2}=\dfrac{6\sqrt{3}-3}{36-3}=\dfrac{-3+6\sqrt{3}}{33}=-\dfrac{3}{33}+\dfrac{6}{33}\sqrt{3}$

$=-\dfrac{1}{11}+\dfrac{2}{11}\sqrt{3}$

Ex-4-9:   (i)   Find the range of values of k for which the equation

$x^2+5x+k=0$ has one or more real roots.

Ans: For real roots: $b^2-4ac>0$

$5^2-4k>0$ $-4k>-25$ $4k<25$ $\Rightarrow k<\dfrac{25}{4}$

(ii)   Solve the equation $4x^2+20x+25=0$

$\Rightarrow x_{1,2}=\dfrac{-b\pm\sqrt{b^2}}{2a}=\dfrac{-20\pm\sqrt{(20)^2-4(4)(25)}}{2(4)}=\dfrac{-20\pm\sqrt{400-400}}{8}$

$=\dfrac{-20\pm\sqrt{0}}{8}=-\dfrac{20}{8}=-\dfrac{5}{2}$ $\Rightarrow x_1=x_2=-\dfrac{5}{2}$

OR factorise the expression:

$4x^2+20x+25=0\Rightarrow(2x+5)(2x+5)=0\Rightarrow x_1=x_2=-\dfrac{5}{2}$

Ex-4-10: A circle has equation $x^2 + y^2 = 45$.

(i) State the centre and radius of this circle.

Ans: The centre is at (0,0) and the radius is $\sqrt{45}$

(ii) The circle intersects the line with equation $x + y = 3$ at two points, A and B. Find algebraically the coordinates of A and B.

Ans: $x^2 + y^2 = 45$

$x + y = 3 \qquad \Rightarrow y = 3 - x$

$x^2 + (3-x)^2 = 45 \quad \Rightarrow x^2 + 9 - 6x + x^2 = 45 \quad \Rightarrow 2x^2 - 6x = 45 - 9 = 36$

$\Rightarrow 2x^2 - 6x - 36 = 0 \quad \Rightarrow x^2 - 3x - 18 = 0 \quad \Rightarrow (x-6)(x+3) = 0$

$\Rightarrow x = 6 \quad and \quad x = -3$

Points: A(6,-3), and B(-3,6)

Show that the distance AB is $\sqrt{162}$

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-3-6)^2 + (6-(-3))^2} = \sqrt{(-9)^2 + (9)^2} = \sqrt{81 + 81} = \sqrt{162}$$

Ex-4-11: (i) Write $x^2 - 7x + 6$ in the form $(x-a)^2 + b$

Ans:

$$x^2 - 7x + 6 = x^2 - 7x + \left(\frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 + 6 = x^2 - 7x + \left(\frac{7}{2}\right)^2 + 6 - \frac{49}{4} = \left(x - \frac{7}{2}\right)^2 -$$

$$+ \frac{24-49}{4} = \left(x - \frac{7}{2}\right)^2 - \frac{25}{4}$$

(ii) State the coordinates of the minimum point on the graph of
$y = x^2 - 7x + 6$

Ans: $y = x^2 - 7x + 6 = \left(x - \frac{7}{2}\right)^2 - \frac{25}{4}$,

the minimum value of $y = -\dfrac{25}{4}$ when $x = \dfrac{7}{2}$, the coordinates of the minimum

point is: $\left(\dfrac{7}{2}, -\dfrac{25}{4}\right)$

(iii) Find the coordinates of the points where the graph of $y = x^2 - 7x + 6$ crosses the axes and sketch the graph.

(iv) Show that the graphs $y = x^2 - 7x + 6$ and $y = x^2 - 3x + 4$ intersect only once. Find the x-coordinate of the point of intersection.

Ans: $y = x^2 - 7x + 6$ and $y = x^2 - 3x + 4$

$\Rightarrow x^2 - 7x + 6 = x^2 - 3x + 4 \Rightarrow x^2 - x^2 - 7x + 3x + 6 - 4 = 0$

$\Rightarrow -4x + 2 = 0 \Rightarrow -4x = -2 \Rightarrow x = \dfrac{1}{2}$ and

$y = x^2 - 7x + 6 = \left(\dfrac{1}{2}\right)^2 - 7\left(\dfrac{1}{2}\right) + 6 = \dfrac{1}{4} - \dfrac{7}{2} + 6 = \dfrac{11}{4}$ Hence point $\left(\dfrac{1}{2}, \dfrac{11}{4}\right)$

Ex-4-12: (i) Sketch the graph of $y = x(x-3)^2$

(ii) Show that the equation $y = x(x-3)^2 - 2$ can be expressed as $x^3 - 6x^2 + 9x - 2 = 0$.

Ans: $y = x(x-3)^2 - 2 = x(x^2 - 6x + 9) - 2 = x^3 - 6x^2 + 9x - 2$

(iii) Show that $x = 2$ is one root of this equation and find the other two roots, expressing your answer in surd form.

Ans: $f(x) = x^3 - 6x^2 + 9x - 2$

$f(2) = 2^3 - 6 \times 2^2 + 9 \times 2 - 2 = 8 - 24 + 18 - 2 = 0$

$$
\begin{array}{r}
x^2 - 4x + 1 \\
x - 2 \enclose{longdiv}{x^3 - 6x^2 + 9x - 2} \\
\underline{-\,x^3 \;+\; 2x^2 } \\
-4x^2 + 9x \\
\underline{-4x^2 + 8x} \\
x - 2 \\
\underline{-\,x \;+\; 2} \\
0
\end{array}
$$

$$f(x) = x^3 - 6x^2 + 9x - 2 = (x-2)\left(x^2 - 4x + 1\right)$$

Hence, the quadratic equation is $x^2 - 4x + 1$

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad\qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$x_1 = \frac{4 + \sqrt{(-4)^2 - 4\times1\times1}}{2\times1} = \frac{4 + \sqrt{16 - 4}}{2} = \frac{4 + \sqrt{12}}{2} = \frac{4 + \sqrt{4\times3}}{2} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$$

$$x_2 = \frac{4 - \sqrt{(-4)^2 - 4\times1\times1}}{2\times1} = \frac{4 - \sqrt{16 - 4}}{2} = \frac{4 - \sqrt{12}}{2} = \frac{4 - \sqrt{4\times3}}{2} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$$

Show the location of these roots on your sketch graph in part(i)

PAPER-5

SECTION A(C1-6-6-06)

Ex-5-1: The volume of a cone is given by the formula $V = \dfrac{1}{3}\pi r^2 h$. Make r the subject of this formula.

Ans: $V = \dfrac{1}{3}\pi r^2 h \quad \Rightarrow r^2 = \dfrac{3V}{\pi h} \quad \Rightarrow r = \pm\sqrt{\dfrac{3V}{\pi h}} = \sqrt{\dfrac{3V}{\pi h}}$

Ex-5-2: One root of the equation $x^3 + ax^2 + 7 = 0$ is $x = -2$. Find the value of a.

Ans: **Because** $x = -2$ is a root, then $(-2)^3 + a(-2)^2 + 7 = 0 \implies 4a = 8 - 7$

$$\implies 4a = 1 \implies a = \frac{1}{4}$$

Ex-5-3: A line has equation $3x + 2y = 6$. Find the equation of the line parallel to this which passes through the point (2,10).

Ans: $3x + 2y = 6 \implies 2y = -3x + 6 \implies y = -\frac{3}{2}x + 3$, the slope of a line which is

parallel to this is: $m = -\frac{3}{2}$,

$$\implies y = -\frac{3}{2}x + b \quad \implies 10 = -\frac{3}{2}(2) + b \implies b = 10 + 3 = 13 \implies y = -\frac{3}{2}x + 13$$

The equation of the line is: $y = -\frac{3}{2}x + 13$

Ex-5-4: Find the coordinates of the point of intersection of the line $y = 3x + 1$ and $x + 3y = 6$

$$y = 3x + 1$$

$$x + 3y = 6$$

$$x + 3y = 6 \implies x + 3(3x + 1) = 6 \implies x + 9x + 3 = 6 \implies 10x = 3 \implies x = \frac{10}{3}$$

And $y = 3x + 1 = 3\left(\frac{10}{3}\right) + 1 = 10 + 1 = 11$

Ex-5-5: Solve the inequality $x^2 + 2x < 3$

Ans: $x^2 + 2x < 3 \implies x^2 + 2x - 3 < 0 \implies (x + 3)(x - 1) < 0$

$(x + 3)(x - 1) < 0$ $(x + 3)$ $(x - 1)$

	x = -3		x = 1	
$(x + 3)$	-		-	0	+
$(x - 1)$	-	0	+		+
$(x + 3)(x - 1)$	+		-		+

Ans: -3 < x < 1

(0,-3)
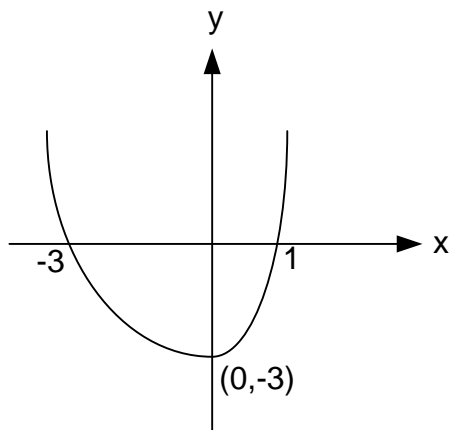
The value of y is positive above the x-axis and the value of y is negative below the x-axis. Hence $-3 < x < 1$ is the answer.

Ex-5-6:   (i)   Simplify $6\sqrt{2} \times 5\sqrt{3} - \sqrt{24}$

Ans: $6\sqrt{2} \times 5\sqrt{3} - \sqrt{24} = 30\sqrt{6} - \sqrt{4 \times 6} = 30\sqrt{6} - \sqrt{4} \times \sqrt{6} = 30\sqrt{6} - 2\sqrt{6} = 28\sqrt{6}$

(ii)   Express $\left(2 - 3\sqrt{5}\right)^2$ in the form $a + b\sqrt{5}$, where **a** and **b** are integers

Ans: $\left(2 - 3\sqrt{5}\right)^2 = 2^2 - 2 \times 2 \times 3\sqrt{3} + \left(3\sqrt{5}\right)^2 = 4 - 12\sqrt{3} + 45 = 49 - 12\sqrt{3}$

Ex-5-7:   Calculate 6_{C_3}

Ans: $6_{C_3} = \dfrac{6!}{(6-3)!3!} = \dfrac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 3 \times 2 \times 1} = \dfrac{720}{36} = 20$

Note: $n_{C_r} = \dfrac{n!}{(n-r)!\, r!}$

Ex-5-8: Find the coefficient of x^3 in the expansion of $(1-2x)^6$

Ans:

$$(a-b)^n = \frac{a^n}{0!} - \frac{n \times a^{n-1}b}{1!} + \frac{n(n-1) \times a^{n-2}b^2}{2!} - \frac{n(n-1)(n-2) \times a^{n-3}b^3}{3!} + \dots - b^n$$

$$\Rightarrow (1-2x)^6 = \frac{1^6}{0!} - \frac{6 \times 1^5(2x)}{1!} + \frac{6 \times 5 \times 1^4 \times (2x)^2}{2!} - \frac{6 \times 5 \times 4 \times 1^3(2x)^3}{3!} + \dots$$

Therefore, the coefficient of x^3 is: $-\dfrac{6 \times 5 \times 4 \times 1^3(2x)^3}{3!} = -\dfrac{960}{6} = -160x^3$

OR the following method is very good, always use this, if n is positive integer!!

$$(a+b)^6 = \binom{6}{0}a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \dots$$

Looking there properly: $(1-2x)^6 = \dots + \binom{6}{3}1^3(-2x)^3 + \dots$

$$\binom{6}{3} = \frac{6!}{(6-3)!\,3!} = \frac{6 \times 5 \times 4 \times 3!}{3! \times 3!} = \frac{120}{6} = 20 \quad \textbf{and} \quad 1^3(-2x)^3 = -8x^3$$

$$\Rightarrow \binom{6}{3}1^3(-2x)^3 = -20 \times 8x^3 = -160x^3 \dots \textbf{ans: -160}$$

Ex-5-9: Simplify the following.

(i) $\dfrac{36^{\frac{1}{2}}}{64^{\frac{2}{3}}} = \dfrac{6}{\left(2^6\right)^{\frac{2}{3}}} = \dfrac{6}{(2)^{6 \times \frac{2}{3}}} = \dfrac{6}{2^4} = \dfrac{6}{16} = \dfrac{3}{8}$

(ii) $\dfrac{15\left(a^3 b^2 c\right)^3}{4a^2 c^5} = \dfrac{15 a^9 b^6 c^3}{4a^2 c^5} = \dfrac{15 a^7 b^6}{4c^2}$

Ex-5-10: Find the coordinates of the points of intersection of the circle $x^2 + y^2 = 36$ and the line $y = 2x$. Give your answer in **surd** form.

$x^2 + y^2 = 36$

$y = 2x.$

$x^2 + y^2 = 36 \ \Rightarrow x^2 + (2x)^2 = 36 \ \Rightarrow x^2 + 4x^2 = 36 \ \Rightarrow 5x^2 = 36 \ \Rightarrow x = \pm\sqrt{\dfrac{36}{5}}$

$\Rightarrow x = \pm\sqrt{\dfrac{36}{5}} = \pm\dfrac{\sqrt{36}}{\sqrt{5}} = \pm\dfrac{6}{\sqrt{5}} = \pm\dfrac{6\sqrt{5}}{\sqrt{5}\times\sqrt{5}} = \pm\dfrac{6\sqrt{5}}{5}$ and $y = 2x = \pm\dfrac{12\sqrt{5}}{5}$

SECTION B (P-5)

Ex-5-11: A(9,8), B(5,0) and C(3,1) are three points.

(i) Show that AB and BC are perpendicular.

Ans: If AB and BC are perpendicular, then their gradients should be $m_1 = -\dfrac{1}{m_2}$. Gradient

of $AB = m_1 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{0 - 8}{5 - 9} = \dfrac{-8}{-4} = 2$

Gradient of $BC = m_2 = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{1 - 0}{3 - 5} = \dfrac{1}{-2} = -\dfrac{1}{2}$. Because $m_1 = -\dfrac{1}{m_2}$, therefore AB and BC are perpendicular.

(ii) Find the equation of the circle with AC as diameter. You need not simplify your answer.

Show that B lies on this circle.

Ans: **Since AC is the diameter of the circle, then** $r = \dfrac{AC}{2}$ and the coordinates of the centre of the circle are the midpoints of the diameter AC:

$$AC = \sqrt{(1-8)^2 + (3-9)^2} = \sqrt{49 + 36} = \sqrt{85}$$

$$r = \frac{AC}{2} = \frac{\sqrt{85}}{2}$$

Midpoint of AC=Centre= $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2} \right) = \left(\dfrac{9+3}{2}, \dfrac{8+1}{2} \right) = (6, 4.5)$

Equation of the circle= $(x-6)^2 + (y-4.5)^2 = \dfrac{85}{4}$

Now to show that B lies on the circle, the coordinates of point B must satisfy the equation.

$$(x-6)^2 + (y-4.5)^2 = 85 \qquad \Rightarrow (5-6)^2 + (0-4.5)^2 = 1 + \frac{81}{4} = \frac{85}{4}$$

(iii)   BD is a diameter of the circle. Find the coordinates of D.

Midpoint of BD=Centre= $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2} \right) = \left(\dfrac{5+x}{2}, \dfrac{0+y}{2} \right) = (6, 4.5)$

$$\Rightarrow \frac{5+x}{2} = 6 \qquad \Rightarrow x+5 = 12 \quad \Rightarrow x = 12-5 = 7 \Rightarrow \frac{0+y}{2} = 4.5 \qquad \Rightarrow y = 9$$

Coordinates of BD= $=(7, 9)$

Ex-5-12:   You are given that $f(x) = x^3 + 9x^2 + 20x + 12$.

(i)   Show that $x = -2$ is a root of $f(x) = 0$.

Ans: If $x = -2$ is a root of $f(x) = x^3 + 9x^2 + 20x + 12 = 0$, then $f(-2) = 0$

$$\Rightarrow f(-2) = (-2)^3 + 9(-2)^2 + 20(-2) + 12 = -8 + 36 - 40 + 12 = 48 - 48 = 0$$

Since $f(-2) = 0$, then $x = -2$, is a root of $f(x) = 0$.

(ii)   Divide $f(x)$ by $x + 6$.

Ans:

$$\begin{array}{r} x^2+3x+2 \\ \hline X+6 \enclose{longdiv}{X^3+9x^2+20x+12} \\ X^3+6x^2 \\ \hline 3x^2+20x \\ 3x^2+18x \\ \hline 2x+12 \\ 2x+12 \\ \hline 0 \end{array}$$

(iii)   Express $f(x)$ in fully factorised form.

Ans: $f(x) = x^3 + 9x^2 + 20x + 12 = (x+6)(x^2 + 3x + 2) = (x+6)(x+2)(x+1)$

(iv)   Sketch the graph of $y = f(x)$.

(v)   Solve the equation $f(x) = 12$.

Ans:

$f(x) = x^3 + 9x^2 + 20x + 12 = 12 \implies x^3 + 9x^2 + 20x = 0 \implies x(x^2 + 9x + 20) = x(x+4)(x+5) = 0$
$\implies x = 0, -4, -5$ are the roots of $f(x) = 12$

Ex-5-13:   The insert shows the graph of $y = \dfrac{1}{x}$, $x \neq 0$.

(i)   Use the graph to find approximate roots of the equation $\dfrac{1}{x} = 2x + 3$, showing your method clearly.
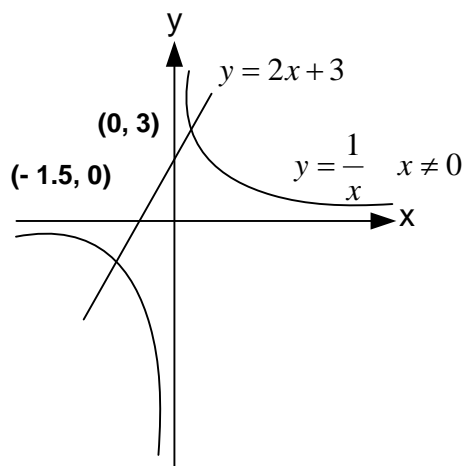
(ii)   Rearrange the equation $\dfrac{1}{x} = 2x + 3$ to form a quadratic equation. Solve the

         resulting equation. Leaving your answers in the form $\dfrac{p \pm \sqrt{q}}{r}$

(iii)   Draw the graph of $y = \dfrac{1}{x} + 2$, $x \neq 0$, on the grid used for part(i).

(iv)   Write down the values of x which satisfy the equation $\dfrac{1}{x} + 2 = 2x + 3$.

PAPER-6

SECTION A (C1-16-1-07)

Ex-6-1:   Find in the form $y = ax + b$, the equation of the line through (5,12) which is parallel to $y = -2x + 5$.

Ans: $y = -2x + 5$, the slope of a line which is parallel to this is: $m = -2$,

         $\Rightarrow y = -2x + b \quad \Rightarrow 12 = -2(5) + b \Rightarrow b = 12 + 10 = 22 \Rightarrow y = -2x + 22$

         The equation of the line is: $y = -2x + 22$

Ex-6-2:   Sketch the graph of $y = 16 - x^2$.

Ans:
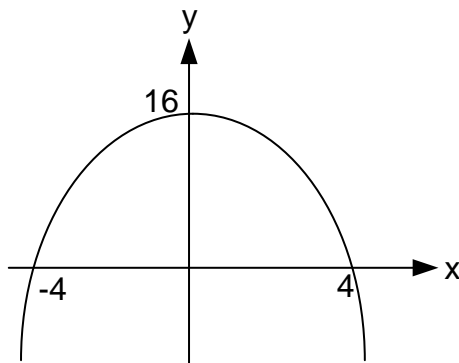
Ex-6-3:   Make a the subject of the equation $2a + 5c = af + 7c$.

Ans: $2a + 5c = af + 7c$ $\Rightarrow 2a - af = 7c - 5c = 2c$ $\Rightarrow a(2 - f) = 2c$

$$\Rightarrow a = \frac{2c}{2 - f}$$

Ex-6-4:   When $x^3 + kx + 5$ is divided by $x - 2$, the remainder is 5. Use the remainder theorem to find the value of k.

Ans: $f(x) = x^3 + kx + 5$ $\Rightarrow f(2) = 2^3 + 2k + 5 = 5$ $\Rightarrow 2k = 5 - 5 - 8 = -8$

$$\Rightarrow k = -4$$

Ex-6-5:   Calculate the coefficient of x^4 in the expansion of $(x + 5)^6$.

Ans: $(x + 5)^6 = x^6 + 6x^5(5) + 15x^4(5)^2 + 20x^3(5)^3 + 15x^2(5)^4 + 6x(5)^5 + (5)^6$

$\Rightarrow (x + 5)^6 = x^6 + 30\ x^5 + 375\ x^4 + 2500\ x^3 + 9375\ x^2 + 18750\ x + 15625$

Therefore, the coefficient of x^4 is 375.

OR the following method is very good, always use this if n is positive integer!!

$$(a + b)^6 = \binom{6}{0}a^6 + \binom{6}{1}a^5 b + \binom{6}{2}a^4 b^2 + \binom{6}{3}a^3 b^3 + \binom{6}{4}a^2 b^4 + \ldots$$

Looking there properly: $(x + 5)^6 = \ldots + \binom{6}{4}x^4(5)^2 + \ldots$

$$\binom{6}{4} = \frac{6!}{(6-4)!\ 4!} = \frac{6 \times 5 \times 4!}{2! \times 4!} = \frac{30}{2} = 15 \quad \text{and} \quad x^4(5)^2 = 25x^4$$

$$\Rightarrow \binom{6}{4}x^4(5)^2 = 15 \times 25x^4 = 375x^4 \textbf{ ans: 375}$$

Ex-6-6:   Find the value of each of the following, giving each answer as an integer or fraction as appropriate.

(i) $25^{\frac{3}{2}} = \left(\sqrt{25}\right)^3 = 5^3 = 125$

(ii) $\left(\dfrac{5}{3}\right)^{-2} = \dfrac{5^{-2}}{3^{-2}} = \dfrac{3^2}{5^2} = \dfrac{9}{25}$

Ex-6-7: You are given that $a = \dfrac{3}{2}$, $b = \dfrac{9 - \sqrt{17}}{4}$, $c = \dfrac{9 + \sqrt{17}}{4}$. Show that

$$a + b + c = \dfrac{3}{2} + \dfrac{9 - \sqrt{17}}{4} + \dfrac{9 + \sqrt{17}}{4} = \dfrac{6 + 9 - \sqrt{17} + 9 + \sqrt{17}}{4} = \dfrac{24}{4} = 6$$

$$abc = \left(\dfrac{3}{2}\right)\left(\dfrac{9 - \sqrt{17}}{4}\right)\left(\dfrac{9 + \sqrt{17}}{4}\right) = \left(\dfrac{3}{2}\right)\left(\dfrac{9^2 - \left(\sqrt{17}\right)^2}{16}\right) = \left(\dfrac{3}{2}\right)\left(\dfrac{81 - 17}{16}\right) = \left(\dfrac{3}{2}\right)\left(\dfrac{64}{16}\right) = \left(\dfrac{3}{2}\right)(4) = \dfrac{12}{2} = 6$$

$$\Rightarrow abc = a + b + c = 6$$

Ex-6-8: Find the set of values of k for which the equation $2x^2 + kx + 2 = 0$ has no real roots.

Ans: The equation $2x^2 + kx + 2 = 0$ has no real roots when

$$b^2 - 4ac < 0 \quad \Rightarrow k^2 - 4(2)(2) < 0 \quad \Rightarrow k^2 - 16 < 0 \quad \Rightarrow k^2 < 16 \Rightarrow -4 < k < 4$$

$-4 < k < 4$
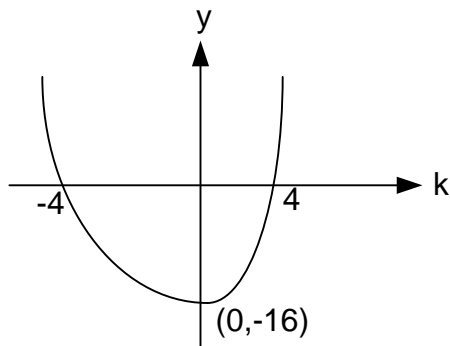
(x − 4) (x + 4) < 0   (x + 4)   (x - 4)

	x = - 4		**x = 4**	
(x - 4)	**-**		**-**	**0** **+**
(x + 4)	**-**	**0**	**+**	**+**
(x − 4)(x + 4)	**+**		**-**	**+**

Ans: **- 4 < x < 4**

Ex-6-9:   (i)   Simplify $3a^3b \times 4(ab)^2$.

Ans: $3a^3b \times 4(ab)^2 = \left(3a^3\right)\left(4a^2b^2\right) = 12\,a^5b^2$

(ii)   Factorise $x^2 - 4$ and $x^2 - 5x + 6$. Hence express $\dfrac{x^2 - 4}{x^2 - 5x + 6}$ as a fraction in its simplest form.

Ans: $\dfrac{x^2 - 4}{x^2 - 5x + 6} = \dfrac{(x-2)(x+2)}{(x-2)(x-3)} = \dfrac{x+2}{x-3}$

Ex-6-10:   Simplify $\left(m^2 + 1\right)^2 - \left(m^2 - 1\right)^2$, showing your method.

Hence, given the right-angled triangle in Fig.10, express p in terms of m , simplifying your answer.

m² - 1

m² + 1

Fig.10

p
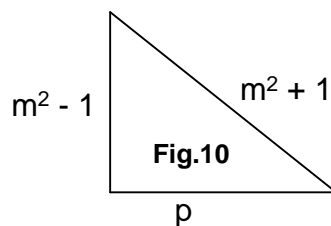
Ans: $\left(m^2 + 1\right)^2 - \left(m^2 - 1\right)^2 = m^4 + 2m^2 + 1 - \left(m^4 - 2m^2 + 1\right) = m^4 + 2m^2 + 1 - m^4 + 2m^2 - 1 = 4m^2$

$p^2 = \left(m^2 + 1\right)^2 - \left(m^2 - 1\right)^2 = 4m^2$

$\Rightarrow p = 2m$

Ex-6-11: The graph of $y = x + \dfrac{1}{x}$ is shown on the insert. The lowest point on one branch is (1,2). The highest point on the other branch is (-1,-2).

(i) Use the graph to solve the following equations, showing your method clearly.

(A) $x + \dfrac{1}{x} = 4$

(B) $2x + \dfrac{1}{x} = 4$

(ii) The equation $(x-1)^2 + y^2 = 4$ represents a circle. Find in exact

form the coordinates of the points of intersection of this circle with the y-axis.

Ans:

When this circle crosses the y-axis,

$x = 0 \implies (0-1)^2 + y^2 = 4 \implies y^2 = 4 - 1 = 3 \implies y = \pm\sqrt{3}$

The coordinates of the points of intersection of this circle with the y-

Axis are: $(0, \sqrt{3})$ and $(0, -\sqrt{3})$

(iii) State the radius and the coordinates of the centre of this circle. Explain how these can be used to deduce from the graph that this circle touches one branch of the curve $y = x + \dfrac{1}{x}$ but does not intersect with the other.

Ex-6-12: Use coordinate geometry to answer this question. Answers obtained from accurate drawing will receive no marks.

A and B are points with coordinates (-1,4) and (7,8) respectively.

(i) Find the coordinates of the midpoint, M, of AB.

Show also that the equation of the perpendicular bisector of AB is $y + 2x = 12$.

Ans: The coordinates of the midpoint, M, of $AB = \left(\dfrac{7-1}{2}, \dfrac{8+4}{2}\right) = (3,6)$

Gradient of $AM = m = \dfrac{8-4}{7-(-1)} = \dfrac{4}{8} = \dfrac{1}{2}$

Hence, gradient of perpendicular bisector is $= m_1 = -\dfrac{1}{m} = -2$

$y = -2x + b \qquad \Rightarrow 6 = -2(3) + b \qquad \Rightarrow b = 6 + 6 = 12$

$y = -2x + 12$

(ii)   Find the area of the triangle bounded by the perpendicular bisector, the y-axis and the line AM, as sketched in Fig.11.
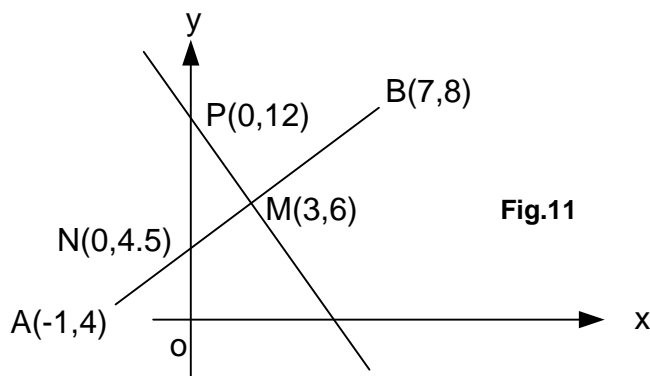
Ans:

Gradient of $AM = m = \dfrac{8-4}{7-(-1)} = \dfrac{4}{8} = \dfrac{1}{2}$

$y = \dfrac{1}{2}x + b \qquad \Rightarrow 4 = \dfrac{1}{2}(-1) + b \qquad \Rightarrow b = 4 + \dfrac{1}{2} = \dfrac{9}{2}$

$y = \dfrac{1}{2}x + \dfrac{9}{2}$ point where it crosses y-axis: $\left(0, \dfrac{9}{2}\right)$. Hence coordinates

of point N is: $\left(0, \dfrac{9}{2}\right) = (0, 4.5)$

Fig.11

$$y = -2x + 12 \quad \Rightarrow P(0,12)$$

$$NM = \sqrt{(3-0)^2 + (6-4.5)^2} = \sqrt{9+2.25} = \sqrt{11.25}$$

$$MP = \sqrt{(0-3)^2 + (12-6)^2} = \sqrt{9+36} = \sqrt{45}$$

Area $= \dfrac{NM \times PM}{2} = \dfrac{\sqrt{11.25} \times \sqrt{45}}{2}$

Ex-6-13: Fig-6-13 shows a sketch of the curve $y = f(x)$, where $f(x) = x^3 - 5x + 2$.
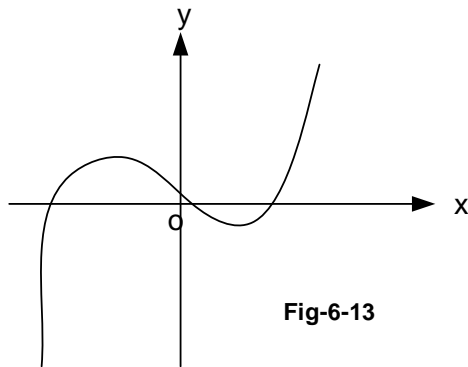
Fig-6-13

(i) Use the fact that $x = 2$ is a root of $f(x) = 0$ to find the exact values of the other two roots of $f(x) = 0$, expressing your answers as simply as possible

```
                    x²+2x-1
          x-2  |  x³-5x+2
                  x³-2x²
                  -   +
                  _____
                  2x²-5x
                  2x²-4x
                  -    +
                  _____
                     -x+2
                     -x+2
                     +  -
                  _____
                      0
```

$$f(x) = x^3 - 5x + 2 = (x-2)(x^2 + 2x - 1)$$

$$\Rightarrow (x^2 + 2x - 1) = 0$$

$$\Rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2}}{2a} = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-1)}}{2(1)} = \frac{-2 \pm \sqrt{4+4}}{2}$$

$$= \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

The roots are: $x_1 = 2$, $x_2 = -1 + \sqrt{2}$, $x_3 = -1 - \sqrt{2}$

(ii)   Show that $f(x-3) = x^3 - 9x^2 + 22x - 10$.

$$f(x) = x^3 - 5x + 2 \quad \Rightarrow f(x-3) = (x-3)^3 - 5(x-3) + 2 = x^3 - 3x^2(3) + 3x(3)^2 - 3^3 - 5x + 15 + 2$$

$$\Rightarrow f(x-3) = x^3 - 9x^2 + 27x - 27 - 5x + 17$$

$$\Rightarrow f(x-3) = x^3 - 9x^2 + 22x - 10$$

(iii)   Write down the roots of $f(x-3) = 0$. **Do it?**

PAPER-7

SECTION A(C1-7-6-07) Calculators: No

Ex-7-1:   Solve the inequality $1 - 2x < 4 + 3x$.

Ans: $1 - 2x < 4 + 3x$

$-2x - 3x < 4 - 1$

$-5x < 3$

$x > -\dfrac{3}{5}$

Ex-7-2:   Make t the subject of the equation $s = \dfrac{1}{2}at^2$.

Ans: $s = \dfrac{1}{2}at^2$

$t^2 = \dfrac{2s}{a}$ $\qquad\qquad \Rightarrow t = \pm\sqrt{\dfrac{2s}{a}} = \sqrt{\dfrac{2s}{a}}$

Ex-7-3: You are given that $f(x) = x^3 + kx + c$. The value of $f(0)$ is 6, and $x - 2$ is a factor of $f(x)$.

Find the values of k and c.

Ans: $f(x) = x^3 + kx + c$

$f(0) = 0^3 + k \times 0 + c = 6 \qquad \Rightarrow c = 6$

$f(2) = 2^3 + k \times 2 + 6 = 0 \qquad \Rightarrow 2k = -6 - 8 = -14 \qquad \Rightarrow k = -7$

Ex-7-4: (i) Find a, given that $a^3 = 64 x^{12} y^3$.

Ans: $a^3 = 64 \ x^{12} y^3 \quad \Rightarrow a = \left(4^3 x^{12} y^3\right)^{\frac{1}{3}} = 4x^4 y$

(ii) Find the value of $\left(\dfrac{1}{3}\right)^{-5} = \dfrac{1^{-5}}{3^{-5}} = \dfrac{3^5}{1^5} = \dfrac{243}{1} = 243$

Ex-7-5: Find the coefficient of x^3 in the expansion of $(3 - 2x)^5$.

Ans: $(a+b)^n = \dfrac{a^n}{0!} + \dfrac{n \times a^{n-1}b}{1!} + \dfrac{n(n-1) \times a^{n-2}b^2}{2!} + \dfrac{n(n-1)(n-2) \times a^{n-3}b^3}{3!} + \ldots + b^n$

$(3-2x)^5 = \dfrac{3^5}{0!} + \dfrac{5 \times 3^4 (-2x)}{1!} + \dfrac{5 \times 4 \times 3^3 (-2x)^2}{2!} + \dfrac{5 \times 4 \times 3 \times 3^2 (-2x)^3}{3!} + \ldots$

$\dfrac{5 \times 4 \times 3 \times 3^2 (-2x)^3}{3!} = -\dfrac{4320}{6} x^3 = -720 x^3$

The coefficient of x^3 is: **-720**

OR the following method is very good, always use this if n is positive integer!!

$(a+b)^6 = \binom{6}{0}a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \ldots$

Looking there properly: $(3-2x)^5 = \ldots + \binom{5}{3}(3)^2(-2x)^3 + \ldots$

$$\binom{5}{3} = \frac{5!}{(5-3)!\,3!} = \frac{5 \times 4 \times 3!}{2! \times 3!} = \frac{20}{2} = 10 \quad \textcolor{red}{\textbf{and}} \quad 3^2 (-2x)^3 = -72x^3$$

$$\Rightarrow \binom{5}{3}(3)^2 (-2x)^3 = -720x^3$$

The coefficient of x^3 is: **-720**

Ex-7-6: Solve the equation $\dfrac{4x+5}{2x} = -3$.

Ans: $\dfrac{4x+5}{2x} = -3 \Rightarrow -6x = 4x+5 \quad \Rightarrow -6x - 4x = 5 \Rightarrow -10x = 5 \Rightarrow x = -\dfrac{5}{10} = -\dfrac{1}{2}$

Ex-7-7: (i) Simplify $\sqrt{98} - \sqrt{50}$

Ans: $\sqrt{98} - \sqrt{50} = \sqrt{49 \times 2} - \sqrt{25 \times 2} = 7\sqrt{2} - 5\sqrt{2} = 2\sqrt{2}$

(ii) Express $\dfrac{6\sqrt{5}}{2+\sqrt{5}}$ in the form $a + b\sqrt{5}$, where **a** and **b** are integers.

Ans: $\dfrac{6\sqrt{5}}{2+\sqrt{5}} = \dfrac{6\sqrt{5} \times (2+\sqrt{5})}{(2+\sqrt{5})(2-\sqrt{5})} = \dfrac{12\sqrt{5} + 6(\sqrt{5})^2}{2^2 - (\sqrt{5})^2} = \dfrac{12\sqrt{5} + 30}{4-5} = -30 - 12\sqrt{5}$

Ex-7-8: (i) A curve has equation $y = x^2 - 4$. Find the x-coordinates of the points on the curve where $y = 21$.

Ans: $y = x^2 - 4 \Rightarrow 21 = x^2 - 4 \quad \Rightarrow x^2 = 25 \quad \Rightarrow x = \pm 5$

(ii) The curve $y = x^2 - 4$ is translated by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Write down an equation for the translated curve. You need not simplify your answer.

Ans: $y = (x-2)^2 - 4$

Ex-7-9: The triangle shown in Fig-7-9 has hight (x+1)cm and base (2x-3)cm. Its area is $9cm^2$.
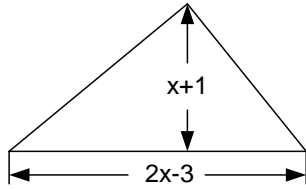
Fig-7-9

(i) Show that $2x^2 - x - 21 = 0$.

Ans: Area $= \dfrac{1}{2}(x+1)(2x-3) = 9 \Rightarrow 2x^2 - 3x + 2x - 3 = 18 \Rightarrow 2x^2 - x - 21 = 0$

(ii) By factorising, solve the equation $2x^2 - x - 21 = 0$. Hence find the height and base of the triangle.

Ans: $2x^2 - x - 21 = 0 \qquad \Rightarrow (2x-7)(x+3) = 0$

$(2x-7) = 0 \Rightarrow x = \dfrac{7}{2} \quad (x+3) = 0 \Rightarrow x = -3(NA)$

Base $= 2x - 3 = 2\left(\dfrac{7}{2}\right) - 3 = 7 - 3 = 4$

Hight $= x + 1 = \dfrac{9}{2}$

SECTION B (P-7)

Ex-7-10: A circle has centre C(1,3) and passes through the point A(3,7) as shown in Fig-7-10.

Fig-7-10

(i) Show that the equation of the tangent at A is $x + 2y = 17$.

Ans: Gradient of CA $= \dfrac{7-3}{3-1} = \dfrac{4}{2} = 2$

54

The gradient of the tangent at A is $=-\dfrac{1}{2}$

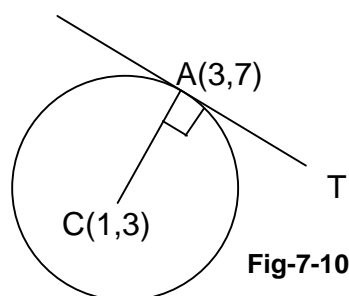The equation of the tangent at A is $y=-\dfrac{1}{2}x+b$ $\quad \Rightarrow 7=-\dfrac{1}{2}(3)+b \quad \Rightarrow b=7+1.5=\dfrac{17}{2}$

The equation of the tangent at A is $y=-\dfrac{1}{2}x+\dfrac{17}{2}$ $\quad \Rightarrow x+2y=17 \quad \Rightarrow OK$

(ii)   The line with equation $y=2x-9$ intersects this tangent at the point T. Find the coordinates of T.

Ans: $y=-\dfrac{1}{2}x+\dfrac{17}{2}$

$y=2x-9=-\dfrac{1}{2}x+\dfrac{17}{2} \quad \Rightarrow 4x-18=-x+17 \quad \Rightarrow 5x=35 \quad \Rightarrow x=7$

$y=2x-9=2(7)-9=14-9=5 \qquad\qquad$ Point T=(7,5)

Ex-7-11:   (i)   Write $4x^2-24x+27$ in the form $a(x-b)^2+c$.

Ans: $4x^2-24x+27=4\left(x^2-6x+\dfrac{27}{4}\right)=4\left(x^2-6x+9-9+\dfrac{27}{4}\right)=4(x-3)^2-9$

(ii)   States the coordinates of the minimum point on the curve
$y=4x^2-24x+27$.

Ans: $4x^2-24x+27=4(x-3)^2-9$

The coordinates of the minimum point on the curve: (3,-9)

(iii)   Solve the equation $4x^2-24x+27=0$.

Ans:

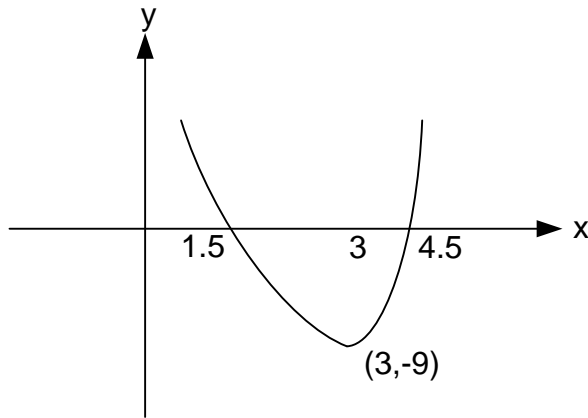$4x^2-24x+27=4(x-3)^2-9=[2(x-3)-3][2(x-3)+3]=0 \quad \Rightarrow (2x-6-3)(2x-6+3)=0$

$\Rightarrow (2x-9)(2x-3)=0 \quad \Rightarrow$

If $(2x-9)=0 \Rightarrow x=\dfrac{9}{2}$ $\qquad and \qquad$ If $(2x-3)=0 \Rightarrow x=\dfrac{3}{2}$

(iv)   Sketch the graph of the curve $y = 4x^2 - 24x + 27$

$$4x^2 - 24x + 27 = 4(x-3)^2 - 9$$

(3,-9)

Ex-7-12:   A cubic polynomial is given by $f(x) = 2x^3 - x^2 - 11x - 12$

(i)   Show that $(x-3)(2x^2 + 5x + 4) = 2x^3 - x^2 - 11x - 12$.
Hence show that $f(x) = 0$ has exactly one real root.

$$(x-3)(2x^2 + 5x + 4) = 2x^3 + 5x^2 + 4x - 6x^2 - 15x - 12 = 2x^3 - x^2 - 11x - 12$$

$$f(x) = (x-3)(2x^2 + 5x + 4) = 0$$

$If \ (x-3) = 0 \ \Rightarrow x = 3$
$and \quad If \ (2x^2 + 5x + 4) = 0 \ \Rightarrow b^2 - 4ac = 25 - 4 \times 2 \times 4 = -7$,

No real roots. Hence only one real root of $x = 3$

(ii)   Show that $x = 2$ is a root of the equation $f(x) = -22$ and find the other roots of this equation.

$$f(x) = 2x^3 - x^2 - 11x - 12 = -22$$

If $x = 2$ is a root of $f(x) = 2x^3 - x^2 - 11x - 12 = -22$, then
$f(2) = 0 \ \Rightarrow 2 \times 2^3 - 2^2 - 11 \times 2 - 12 = 16 - 4 - 22 - 12 = -22 \ \Rightarrow OK$

(iii)   Using the results from the previous parts, sketch the graph of $y = f(x)$.

PAPER-8

Ex-8-1:   Make v the subject of the equation $E = \dfrac{1}{2}mv^2$.

Ans: $E = \dfrac{1}{2}mv^2 \Rightarrow v^2 = \dfrac{2E}{m} \Rightarrow v = \pm\sqrt{\dfrac{2E}{m}}$

Ex-8-2:   Factorise and hence simplify $\dfrac{3x^2 - 7x + 4}{x^2 - 1}$

Ans: $\dfrac{3x^2 - 7x + 4}{x^2 - 1} = \dfrac{(3x-4)(x-1)}{(x+1)(x-1)} = \dfrac{3x-4}{x+1}$

Ex-8-3:   (i)   Write down the value of $\left(\dfrac{1}{4}\right)^0$.

Ans: $\left(\dfrac{1}{4}\right)^0 = 1$

    (ii)   Find the value of $16^{-\frac{3}{2}}$

Ans: $16^{-\frac{3}{2}} = \left(2^4\right)^{-\frac{3}{2}} = 2^{-6} = \dfrac{1}{2^6} = \dfrac{1}{64}$

Ex-8-4:   Find algebraically, the coordinates of the point of intersection of the lines $y = 2x - 5$ and $6x + 2y = 7$.

Ans: $y = 2x - 5$

$6x + 2y = 7 \Rightarrow 6x + 2(2x - 5) = 7 \Rightarrow 6x + 4x - 10 = 7 \Rightarrow 10x = 17 \Rightarrow x = \dfrac{17}{10}$

$y = 2x - 5 = 2\left(\dfrac{17}{10}\right) - 5 = \dfrac{17}{5} - 5 = -\dfrac{8}{5}$ $\Rightarrow x = \dfrac{17}{10}$ **and** $y = -\dfrac{8}{5}$

Ex-8-5:   **(i)**   Find the gradient of the line $4x+5y=24$.

Ans: $4x+5y=24 \qquad \Rightarrow 5y=24-4x \qquad \Rightarrow y=-\dfrac{4}{5}x+\dfrac{24}{5}$

Gradient $m=-\dfrac{4}{5}$

(ii)   A line parallel to $4x+5y=24$ passes through the point (0,12). Find the coordinates of its point of intersection with the x-axis.

Ans: $4x+5y=24 \Rightarrow 5y=-4x+24 \Rightarrow y=-\dfrac{4}{5}x+\dfrac{24}{5}$

$y=mx+b=-\dfrac{4}{5}x+b \Rightarrow 12=b$

$y=-\dfrac{4}{5}x+12$

The intersection with x-axis happens when y=0, $-\dfrac{4}{5}x+12=0 \Rightarrow x=-\dfrac{60}{4}=-15$

Point of intersection with x-axis is: $\left(-15,0\right)$

Ex-8-6:   When x^3+kx+7 is divided by $(x-2)$, the remainder is 3. Find the value of k.

Ans: $f(x)=x^3+kx+7 \qquad \Rightarrow f(2)=2^3+2k+7=3 \Rightarrow 2k=3-7-8=-12$

$\Rightarrow k=-6$

Ex-8-7:   **(i)**   Find the value of 8_{C_3}.

Ans: $8_{C_3}=\dfrac{8\,!}{(8-3)!\,3!}=\dfrac{8\times7\times6\times5!}{5!3!}=56$

(ii)   Find the coefficient of x^3 in the binomial expansion of $\left(1-\dfrac{1}{2}x\right)^8$.

Ans:

Ans: $(a+b)^n = \dfrac{a^n}{0!}+\dfrac{n\times a^{n-1}b}{1!}+\dfrac{n(n-1)\times a^{n-2}b^2}{2!}+\dfrac{n(n-1)(n-2)\times a^{n-3}b^3}{3!}+\ldots +b^n$

$$\left(1-\dfrac{1}{2}x\right)^8 = \dfrac{1^8}{0!}+\dfrac{8\times 1^7\times\left(-\dfrac{1}{2}x\right)}{1!}+\dfrac{8\times 7\times 1^6\times\left(-\dfrac{1}{2}x\right)^2}{2!}+\dfrac{8\times 7\times 6\times 1^5\times\left(-\dfrac{1}{2}x\right)^3}{2!}+\ldots$$

$$\dfrac{8\times 7\times 6\times 1^5\times\left(-\dfrac{1}{2}x\right)^3}{2!}=\dfrac{336}{2}\times\left(-\dfrac{x^3}{8}\right)=-21\,x^3$$

The coefficient of x^3 is: **-21**

OR the following method is very good, always use this in n is positive integer!!

$$(a+b)^8 = \binom{8}{0}a^8 + \binom{8}{1}a^7b + \binom{8}{2}a^6b^2 + \binom{8}{3}a^5b^3 +\ldots$$

Looking there properly: $\left(1-\dfrac{1}{2}x\right)^8 = \ldots +\binom{8}{3}(1)^5\left(-\dfrac{1}{2}x\right)^3 +\ldots$

$$\binom{8}{3}=\dfrac{8!}{(8-3)!\,3!}=\dfrac{8\times 7\times 6\times 5!}{5!\times 3!}=56 \quad\textbf{and}\quad (1)^5\left(-\dfrac{1}{2}x\right)^3 = -\dfrac{x^3}{8}$$

$$\Rightarrow \binom{8}{3}(1)^5\left(-\dfrac{1}{2}x\right)^3 = 56\times\left(-\dfrac{x^3}{8}\right)=-7x^3$$

The coefficient of x^3 is: **-7**

Ex-8-8:   (i)   Write $\sqrt{48}+\sqrt{3}$ in the form $a\sqrt{b}$, where **a** and **b** are integers

and b is as small as possible.

Ans: $\sqrt{48}+\sqrt{3}=\sqrt{16\times 3}+\sqrt{3}=4\sqrt{3}+\sqrt{3}=5\sqrt{3}$

(ii)　Simplify $\dfrac{1}{5+\sqrt{3}}+\dfrac{1}{5-\sqrt{3}}$

Ans: $\dfrac{1}{5+\sqrt{3}}+\dfrac{1}{5-\sqrt{3}}=\dfrac{5-\sqrt{3}+5+\sqrt{3}}{\left(5+\sqrt{3}\right)\left(5-\sqrt{3}\right)}=\dfrac{10}{\left(5^2-\left(\sqrt{3}\right)^2\right)}=\dfrac{10}{25-3}=\dfrac{10}{22}=\dfrac{5}{11}$

Ex-8-9:　　(i)　　Prove that 12 is a factor of $3n^2+6n$ for all even positive integer n.

Ans:　$3n^2+6n=3n(n+2)=\begin{cases}24 & n=2\\72 & n=4\\144 & n=6\\240 & n=8\end{cases}$

SECTION B(P-8)

Ex-8-10:　　Fig-8-10　shows a sketch of the graph of $y=\dfrac{1}{x}$.

Fig-8-10
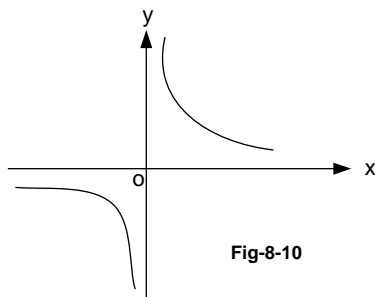
(i)　Sketch the graph of $y=\dfrac{1}{x-2}$, showing clearly the coordinates of any points where it crosses the axes.

(ii)　Find the value of x for which $\dfrac{1}{x-2}=5$

(iii)　Find the x-coordinates of the points of intersection of the graphs of $y=x$ and $y=\dfrac{1}{x-2}$. Give your answers in the form $a\pm\sqrt{b}$.

Show the position of these points on your graph in part(i)

Ex-8-11:   (i)   Write x^2-5x+8 in the form $(x-a)^2+b$ and hence show

that $x^2-5x+8>0$ for all values of x.

Ans: $x^2-5x+8=x^2-5x+\left(\dfrac{5}{2}\right)^2-\left(\dfrac{5}{2}\right)^2+8=\left(x-\dfrac{5}{2}\right)^2-\dfrac{25}{4}+8=\left(x-\dfrac{5}{2}\right)^2+\dfrac{7}{4}$

(ii)   Sketch the graph of $y=x^2-5x+8$, showing the coordinates of the turning point.
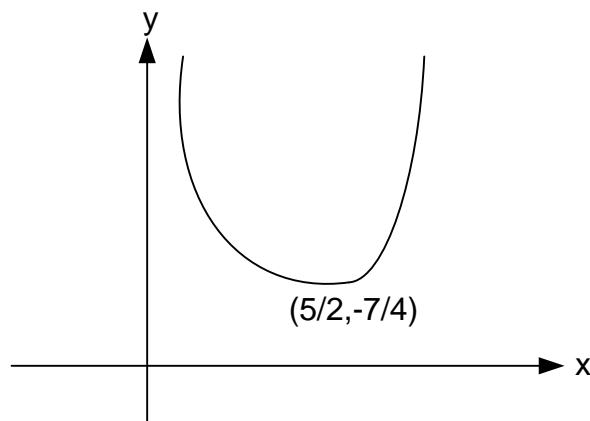
(5/2,-7/4)

Ans: Coordinates of the turning point: $\left(\dfrac{5}{2},\dfrac{7}{4}\right)$

(iii)   Find the set of values of x for which $x^2-5x+8>14$

Ans: $x^2-5x+8>14 \;\Rightarrow x^2-5x+8-14>0 \qquad \Rightarrow x^2-5x-6>0$
$\Rightarrow (x-6)(x+1)>0$

(0,-6)

Ans: $6<x<-1$

(iv)   If $f(x)=x^2-5x+8$, does the graph of $y=f(x)-10$ cross the x-axis? Show how you decide.
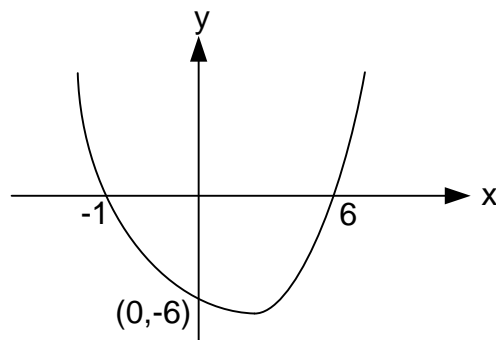
Ans: $f(x)=x^2-5x+8$

$$y=f(x)-10=x^2-5x+8-10=x^2-5x-2$$

Now in order to cross the x-axis, y should be equal to zero.

$$x^2-5x-2=0$$

$$\Rightarrow x_{1,2}=\frac{-b\pm\sqrt{b^2}}{2a}=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(-2)}}{2(1)}=\frac{5\pm\sqrt{25+8}}{2}$$

$$=\frac{5\pm\sqrt{33}}{2}\quad\Rightarrow x_1=\frac{5+\sqrt{33}}{2},\quad and\quad x_2=\frac{5-\sqrt{33}}{2}$$

Ex-8-12:   A circle has equation $x^2+y^2-8x-4y=9$

(i)   Show that the centre of this circle is C(4,2) and find the radius of the circle.

Ans: $x^2+y^2-8x-4y=9\quad\Rightarrow x^2-8x+16-16+y^2-4y+4-4=9$

$$\Rightarrow(x-4)^2+(y-2)^2=29$$

The centre is at: (4,2) and the radius is: $r=\sqrt{29}$

(ii)   Show that the origin lies inside the circle.

(iii)   Show that AB is a diameter of the circle, where A has coordinates (2,7) and B has coordinates (6,-3).

Ans: If AB is a diameter, then centre should be the Midpoints of AB.

Mid Point$\left(\frac{2+6}{2},\frac{7-3}{2}\right)=(4,2)\quad\Rightarrow OK$

(iv)   Find the equation of the tangent to the circle at A. Give your answer in the form $y=mx+c$.

The gradient of AB$=m=\dfrac{-3-7}{6-2}=-\dfrac{10}{4}=-\dfrac{5}{2}$

Hence, the gradient of the tangent is$=-\dfrac{1}{m}=\dfrac{2}{5}$

The equation of the tangent is:

$$y = \frac{2}{5}x + b \quad \Rightarrow 7 = \left(\frac{2}{5}\right)(2) + b \quad \Rightarrow b = 7 - \frac{4}{5} = \frac{31}{7}$$

$$y = \frac{2}{5}x + \frac{31}{7}$$

PAPER-9

Ex-9-1: Solve the inequality $3x - 1 > 5 - x$.

Ans: $3x - 1 > 5 - x$

$3x + x > 5 + 1$

$4x > 6$

$x > \frac{6}{4}$

$x > \frac{3}{2}$

Ex-9-2: (i) Find the points of intersection of the line $2x + 3y = 12$ with the axes.

Ans: The line intersect the x-axis when y=0, $2x + 3(0) = 12 \Rightarrow x = 6$

The line intersect the y-axis when x=0, $2(0) + 3y = 12 \Rightarrow y = 4$

Points: (6,0) and (0,4)

(ii) Find the gradient of this line.

Ans: $2x + 3y = 12 \Rightarrow 3y = -2x + 12 \Rightarrow y = -\frac{2}{3}x + 4$

The gradient of the line is: $m = -\frac{2}{3}$

Ex-9-3: (i) Solve the equation $2x^2 + 3x = 0$.

Ans: $2x^2 + 3x = 0 \Rightarrow 2x(x+3) = 0 \Rightarrow x = 0 \quad and \quad x = -3$

(ii) Find the set of values of k for which the equation $2x^2 + 3x - k = 0$ has no real roots.

The equation $2x^2 + 3x - k = 0$ has no real roots when $b^2 - 4ac < 0$, hence

$$3^2 - 4(2)(-k) < 0 \Rightarrow 8k < -9 \Rightarrow k < -\frac{9}{8}$$

Ex-9-4: Given that n is a positive integer, write down whether the following statements are always true (T), always false (F) or could be either true or false (E).

(i) $2n+1$ is an odd integer (T)

Ans: When $n = Odd$, then $2n+1 = Even + Odd = Odd$ is always ODD

(ii) $3n+1$ is an even integer (E)

Ans: When $n = Odd$, then $3n+1$ is always EVEN

When $n = Even$, then $3n+1$ is always ODD

(iii) n is odd $\Rightarrow n^2$ is odd (T)

Ans: When $n = Odd$, then $n^2 = n \times n = Odd \times Odd = Odd$ is always ODD

(iii) n^2 is odd $\Rightarrow n^3$ is even. (F)

Ans: When $n^2 = Odd$, then $n = Odd$. Hence $n^3 = n \times n \times n = Odd \times Odd \times Odd = Odd$ is always ODD

Ex-9-5: Make x the subject of the equation $y = \dfrac{x+3}{x-2}$.

Ans: $y = \dfrac{x+3}{x-2} \Rightarrow yx - 2y = x+3 \Rightarrow yx - x = 3 + 2y \Rightarrow x(y-1) = 3 + 2y$

$$\Rightarrow x = \frac{2y+3}{y-1}$$

Ex-9-6:   (i)   Find the value of $\left(\dfrac{1}{25}\right)^{-\frac{1}{2}}$

Ans: $\left(\dfrac{1}{25}\right)^{-\frac{1}{2}} = \dfrac{1}{\left(\dfrac{1}{25}\right)^{\frac{1}{2}}} = \dfrac{(25)^{\frac{1}{2}}}{1} = \sqrt{25} = 5$

(ii)   Simplify $\dfrac{\left(2x^2y^3z\right)^5}{4y^2z}$

Ans: $\dfrac{\left(2x^2y^3z\right)^5}{4y^2z} = \dfrac{2^5 x^{10}y^{15}z^5}{4y^2z} = 8x^{10}y^{13}z^4$

Ex-9-7:   (i)   Express $\dfrac{1}{5+\sqrt{3}}$ in the form $\dfrac{a+b\sqrt{3}}{c}$, where **a**, **b** and **c** are integers.

Ans: $\dfrac{1}{5+\sqrt{3}} = \dfrac{5-\sqrt{3}}{\left(5+\sqrt{3}\right)\left(5-\sqrt{3}\right)} = \dfrac{5-\sqrt{3}}{5^2 - 3} = \dfrac{5-\sqrt{3}}{22}$

Ex-9-8:   Find the coefficient of x^3 in the binomial expansion of $(5-2x)^5$.

Ans: $(a+b)^n = \dfrac{a^n}{0!} + \dfrac{n \times a^{n-1}b}{1!} + \dfrac{n(n-1)\times a^{n-2}b^2}{2!} + \dfrac{n(n-1)(n-2)\times a^{n-3}b^3}{3!} + \ldots + b^n$

$(5-2x)^5 = \dfrac{5^5}{0!} + \dfrac{5\times 5^4\left(-2x\right)}{1!} + \dfrac{5\times 4\times 5^3\left(-2x\right)^2}{2!} + \dfrac{5\times 4\times 3\times 5^2\left(-2x\right)^3}{3!} + \ldots$

$\dfrac{5\times 4\times 3\times 5^2\left(-2x\right)^3}{3!} = -\dfrac{1500}{6}x^3 = -2000x^3$

The coefficient of x^3 is: **-2000**

OR the following method is very good, always use this if n is positive integer!!

$$(a+b)^5 = \binom{5}{0}a^5 + \binom{5}{1}a^4 b + \binom{5}{2}a^3 b^2 + \binom{5}{3}a^2 b^3 + \ldots$$

Looking there properly: $(5-2x)^5 = \ldots + \binom{5}{3}(5)^2(-2x)^3 + \ldots$

$$\binom{5}{3} = \frac{5!}{(5-3)!\,3!} = \frac{5 \times 4 \times 3!}{2! \times 3!} = \frac{20}{2} = 10 \quad \textbf{and} \quad 5^2(-2x)^3 = -200x^3$$

$$\Rightarrow \binom{5}{3}(5)^2(-2x)^3 = -2000x^3$$

The coefficient of x^3 is: **-2000**

Ex-9-9:   Solve the equation $y^2 - 7y + 12 = 0$.

Hence solve equation $x^4 - 7x^2 + 12 = 0$.

Ans: $y^2 - 7y + 12 = 0 \quad (y-3)(y-4) = 0$

$\Rightarrow y = 3$ and $y = 4$

Let $y = x^2$, then $x^4 - 7x^2 + 12 = y^2 - 7y + 12 = 0$

When $x^2 = y = 3 \Rightarrow x = \pm\sqrt{3}$

When $x^2 = y = 4 \Rightarrow x = \pm\sqrt{4} = \pm 2$

Ex-9-10:   (i)   Express $x^2 - 6x + 2$ in the form $(x-a)^2 + b$

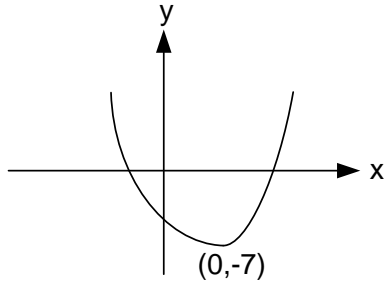**Ans:** $x^2 - 6x + 2 = x^2 - 6x + 9 - 9 + 2 = (x-3)^2 - 7$

(ii)   State the coordinates of the minimum point on the graph of
$y = x^2 - 6x + 2$

Ans: $x^2 - 6x + 2 = (x-3)^2 - 7$

The coordinates of the minimum point is: (3,-7)

(iii)   Sketch the graph of $y = x^2 - 6x + 2$. You need not state the coordinates of the points where the graph intersects the x-axis.

Ans: $x^2 - 6x + 2 = (x-3)^2 - 7$

(0,-7)

(iv)   Solve the simultaneous equations $y = x^2 - 6x + 2$ and $y = 2x - 14$. Hence, show that the line intersect only once. Find the x-coordinate of the line $y = 2x - 14$ is a tangent to the curve $y = x^2 - 6x + 2$.

Ex-9-11:   You are given that $f(x) = 2x^3 + 7x^2 - 7x - 12$.

(i)   Verify that $x = -4$ is a root of $f(x) = 0$.

Ans: If $x = -4$ is a root of $f(x) = 2x^3 + 7x^2 - 7x - 12$, then $f(-4) = 0$

$\Rightarrow f(-4) = 2 \times (-4)^3 + 7 \times (-4)^2 - 7 \times (-4) - 12 = -128 + 112 + 28 - 12 = 0 \Rightarrow OK$

(ii)   Hence express $f(x)$ in fully factorised form.

$$
\begin{array}{r}
2x^2\text{-x-3} \\
\hline
x{+}4 \,\big|\; 2x^3{+}7x^2{-}7x{-}12 \\
2x^3{+}8x^2 \\
\hline
\text{-}x^2\text{-}7x \\
\text{-}x^2\text{-}4x \\
\hline
\text{-}3x{-}12 \\
\text{-}3x{-}12 \\
\hline
0
\end{array}
$$

Ans: $f(x) = 2x^3 + 7x^2 - 7x - 12 = (x-4)(2x^2 - x - 3) = (x-4)(2x-3)(x+1)$

(iii)   Sketch the graph of $y = f(x)$.

(iv)   Show that $f(x-4) = 2x^3 - 17x^2 + 33x$.

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$$f(x) = 2x^3 + 7x^2 - 7x - 12$$

$$\Rightarrow f(x-4) = 2(x-4)^3 + 7(x-4)^2 - 7(x-4) - 12$$
$$= 2\left[x^3 - 3x^2(4) + 3x(4)^2 - 4^3\right] + 7(x^2 - 8x + 16) - 7x + 28 - 12$$
$$= 2x^3 - 24x^2 + 96x - 128 + 7x^2 - 56x + 112 - 7x + 28 - 12$$
$$= 2x^3 - 17x^2 + 33x \quad \Rightarrow OK$$

Ex-9-12: (i) Find the equation of the line passing through A(-1,1) and B(3,9).

Ans: $y = mx + b$ and $m = \dfrac{9-1}{3-(-1)} = \dfrac{8}{4} = 2$

$y = mx + b = 2x + b \qquad \Rightarrow 9 = 2(3) + b \quad \Rightarrow b = 9 - 6 = 3$

$y = 2x + 3$

(ii) Show that the equation of the perpendicular bisector of AB is $2y + x = 11$

Mid-Point of AB $= \left(\dfrac{-1+3}{2}, \dfrac{1+9}{2}\right) = (1,5)$

The gradient of the perpendicular bisector is $m = -\dfrac{1}{2}$

The equation is: $y = -\dfrac{1}{2}x + b \quad \Rightarrow 5 = -\dfrac{1}{2}(1) + b \quad \Rightarrow b = 5 + \dfrac{1}{2} = \dfrac{11}{2}$

The equation is: $y = -\dfrac{1}{2}x + \dfrac{11}{2}$

(iii) A circle has centre (5,3), so that its equation is $(x-5)^2 + (y-3)^2 = k$. Given that the circle passes through A, show that $k = 40$. Show that the circle also passes through B.

Ans: If the circle $(x-5)^2 + (y-3)^2 = k$, passes through A, then

$(-1-5)^2 + (1-3)^2 = k \quad \Rightarrow k = 6^2 + 2^2 = 36 + 4 = 40$

(iv)   Find the x-coordinates of the points where this circle crosses the x-axis.
        Give your answers in surd form.

Ans: The x-coordinates of the points where this circle crosses the x-axis is:

$$(x-5)^2 + (0-3)^2 = 40 \implies (x-5)^2 = 40-9 = 31 \implies x-5 = \pm\sqrt{31}$$

$$\implies x = 5 \pm \sqrt{31}$$

PAPER-10

SECTION A(C1-9-1-09) Calculator: No

Ex-10-1:   State the value of each of the following.

(i) 2^{-3}

Ans: $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$

(ii) 9^0

Ans: $9^0 = 1$

Ex-10-2:   Find the equation of the line passing through (-1,-9) and (3,11). Give your
answer in the form $y = mx + c$.

Ans: Gradient of the line $m = \dfrac{11-(-9)}{3-(-1)} = \dfrac{20}{4} = 5$

$y = 5x + b \qquad \implies 11 = 5(3) + b \implies b = 11 - 15 = -4$

$y = 5x - 4$

Ex-10-3:   Solve the inequality $7 - x < 5x - 2$.

Ans: $7 - x < 5x - 2.$

$-x - 5x < 5x - 2 - 7$

$$-6x < -9$$

$$x > \frac{9}{6}$$

$$x > \frac{3}{2}$$

Ex-10-4: You are given that $f(x) = x^4 + ax - 6$ and that $x - 2$ is a factor of $f(x)$. Find the value of a.

Ans: $f(x) = x^4 + ax - 6$

If $x - 2$ **is a factor of** $f(x) = x^4 + ax - 6$, **then** $f(2) = 2^4 + a(2) - 6 = 0$

$$\Rightarrow 2a = 6 - 16 = -10 \qquad \Rightarrow a = -5$$

Ex-10-5: (i) Find the coefficient of x^3 in the expansion of $(x^2 - 3)(x^3 + 7x + 1)$.

Ans: $(x^2 - 3)(x^3 + 7x + 1) = ... + 7x^3 - 3x^3 + ...$

The coefficient of x^3 is: **4**

(ii) Find the coefficient of x^2 in the binomial expansion of $(1 + 2x)^7$.

Ans: $(a+b)^n = \dfrac{a^n}{0!} + \dfrac{n \times a^{n-1}b}{1!} + \dfrac{n(n-1) \times a^{n-2}b^2}{2!} + \dfrac{n(n-1)(n-2) \times a^{n-3}b^3}{3!} + ... + b^n$

$$(1+2x)^7 = \frac{1^7}{0!} + \frac{7 \times 1^6 (2x)}{1!} + \frac{7 \times 6 \times 1^5 (2x)^2}{2!} + ...$$

$$\Rightarrow \frac{7 \times 6 \times 1^5 (2x)^2}{2!} = \frac{168}{2}x^2 = 84x^2$$

The coefficient of x^2 is: **84**

OR the following method is very good, always use this if n is positive integer!!

$$(a+b)^7 = \binom{7}{0}a^7 + \binom{7}{1}a^6 b + \binom{7}{2}a^5 b^2 + ...$$

Looking there properly: $(1+2x)^7 = ... + \binom{7}{2}(1)^5(2x)^2 + ...$

$\binom{7}{2} = \dfrac{7!}{(7-2)!\,2!} = \dfrac{7 \times 6 \times 5!}{5! \times 2!} = \dfrac{42}{2} = 21$ **and** $1^5(2x)^2 = 4x^2$

$\Rightarrow \binom{7}{2}(1)^5(2x)^2 = 84x^2$

The coefficient of x^2 is: **84**

Ex-10-6:   Solve the equation $\dfrac{3x+1}{2x} = 4$

Ans: $\dfrac{3x+1}{2x} = 4 \Rightarrow 8x = 3x+1 \Rightarrow 8x - 3x = 1 \Rightarrow 5x = 1 \Rightarrow x = \dfrac{1}{5} = 0.2$

Ex-10-7:   (i)   Express $125\sqrt{5}$ in the form 5^k.

Ans: $125\sqrt{5} = 5^3 \times 5^{\frac{1}{2}} = 5^{\frac{7}{2}}$

(ii)   Simplify $\left(4a^3b^5\right)^2$.

Ans: $\left(4a^3b^5\right)^2 = 4^2 a^6 b^{10} = 16\, a^6\, b^{10}$

Ex-10-8:   Find the range of values of k for which the equation $2x^2 + kx + 18 = 0$ does not have real roots.

Ans: The equation $2x^2 + kx + 18 = 0$ has no real roots when

$b^2 - 4ac < 0 \Rightarrow k^2 - 4 \times 2 \times 18 < 0 \Rightarrow k^2 - 144 < 0$
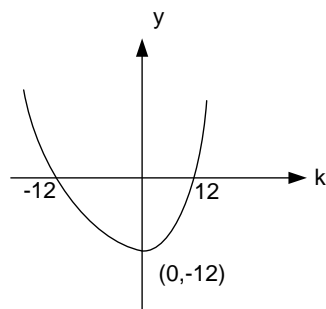
$$-12 < k < 12$$

Ex-10-9: Rearrange $y + 5 = x(y + 2)$ to make y the subject of the formula.

Ans: $y + 5 = x(y + 2)$

$\Rightarrow y - xy = 2x - 5$

$\Rightarrow y(1 - x) = 2x - 5$

$\Rightarrow y = \dfrac{2x - 5}{1 - x}$

Ex-10-10: (i) Express $\sqrt{75} + \sqrt{48}$ in the form $a\sqrt{3}$.

Ans: $\sqrt{75} + \sqrt{48} = \sqrt{3 \times 25} + \sqrt{3 \times 16} = 5\sqrt{3} + 4\sqrt{3} = 9\sqrt{3}$

(ii) Express $\dfrac{14}{3 - \sqrt{2}}$ in the form $b + c\sqrt{d}$.

Ans: $\dfrac{14}{3 - \sqrt{2}} = \dfrac{14(3 + \sqrt{2})}{(3 - \sqrt{2})(3 + \sqrt{2})} = \dfrac{42 + 14\sqrt{2}}{9 - 2} = \dfrac{42 + 14\sqrt{2}}{7} = \dfrac{42}{7} + \dfrac{14\sqrt{2}}{7} = 6 + 2\sqrt{2}$

Ex-10-11: Fig-10-11 shows the points A and B, which have coordinates (-1,0) and (11,4) respectively.

Fig-10-11
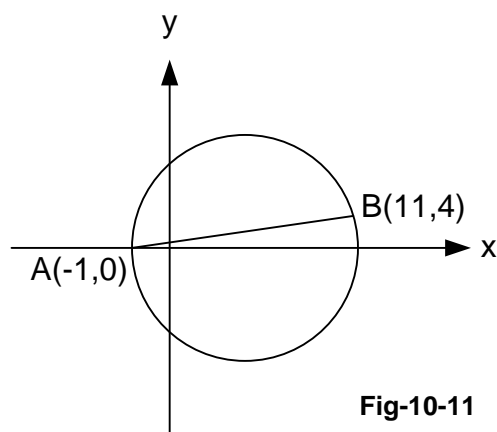
(i) Show that the equation of the circle with AB as diameter may be written as:
$(x - 5)^2 + (y - 2)^2 = 40$

Ans: $r = \dfrac{AB}{2}$ $\Rightarrow AB = \sqrt{(11-(-1))^2 + (4-0)^2} = \sqrt{144+16} = \sqrt{160}$

$r = \dfrac{AB}{2} = \dfrac{\sqrt{160}}{2}$

The coordinates of the centre is the Midpoint of AB

$C\left(\dfrac{11-1}{2}, \dfrac{4+0}{2}\right) = (5, 2)$

Equation of the circle is: $(x-5)^2 + (y-2)^2 = r^2 = \left(\dfrac{\sqrt{160}}{2}\right)^2 = \dfrac{160}{4} = 40$

Hence, equation of the circle is: $(x-5)^2 + (y-2)^2 = 40$

(ii)   Find the coordinates of the points of intersection of this circle with the y-axis. Give your answer in the form $a \pm \sqrt{b}$.

Ans: The intersection of this circle with the y-axis happens when x=0,

$\Rightarrow (0-5)^2 + (y-2)^2 = 40$ $\Rightarrow 25 + (y-2)^2 = 40$ $\Rightarrow (y-2)^2 = 15$

$\Rightarrow (y-2) = \pm\sqrt{15}$ $\Rightarrow y = 2 \pm \sqrt{15}$

Point of intersection with y-axis: $\left(0, 2 \pm \sqrt{15}\right)$

(iii)   Find the equation of the tangent to the circle at B. Hence find the coordinates of the points of intersection of this tangent with the axes.

Ans: Gradient of AB $= \dfrac{11-(-1)}{4-0} = \dfrac{12}{4} = 3$

The gradient of the tangent at B is $= -\dfrac{1}{3}$

The equation of the tangent at b is $y = -\dfrac{1}{3}x + b$ $\Rightarrow 4 = -\dfrac{1}{3}(11) + b$ $\Rightarrow b = 4 + \dfrac{11}{3} = \dfrac{23}{3}$

The equation of the tangent at b is $y = -\dfrac{1}{3}x + \dfrac{23}{3}$

The intersection of this tangent with the x-axis happens when y=0.

$$y = -\frac{1}{3}x + \frac{23}{3} \Rightarrow -\frac{1}{3}x + \frac{23}{3} = 0 \Rightarrow x = 23$$. Hence point (23,0)

The intersection of this tangent with the y-axis happens when x=0.

$$y = -\frac{1}{3}x + \frac{23}{3} \Rightarrow y = 0 + \frac{23}{3} \Rightarrow y = \frac{23}{3}$$. Hence point $\left(0, \dfrac{23}{3}\right)$.

Ex-10-12:   (i)   Find algebraically the coordinates of the points of intersection

of the curve $y = 3x^2 + 6x + 10$ and the line $y = 2 - 4x$.

Ans: $y = 3x^2 + 6x + 10$

$y = 2 - 4x$

$$\Rightarrow 3x^2 + 6x + 10 = 2 - 4x \quad \Rightarrow 3x^2 + 6x + 10 - 2 + 4x = 0 \quad \Rightarrow 3x^2 + 10x + 8 = 0$$

$$3x^2 + 10x + 8 = 0 \quad \Rightarrow (3x + 4)(x + 2) = 0$$

If $(3x + 4) = 0 \Rightarrow x = -\dfrac{4}{3}$ and $y = 2 - 4x = 2 - \left(-\dfrac{4}{3}\right) = \dfrac{10}{3}$

If $(x + 2) = 0 \Rightarrow x = -2 \Rightarrow y = 2 - 4x = 2 - 4(-2) = 2 + 8 = 10$

Points of intersection: $\left(\dfrac{4}{3}, \dfrac{10}{3}\right)$ and $(-2, 10)$

(ii)   Write $3x^2 + 6x + 10$ in the form $a(x + b)^2 + c$

Ans: $3x^2 + 6x + 10 = 3\left(x^2 + 2x + \dfrac{10}{3}\right) = 3\left(x^2 + 2x + 1 - 1 + \dfrac{10}{3}\right) = 3\left[(x + 1)^2 + \dfrac{7}{3}\right] = (x + 1)^2 + 7$

(iii)   Hence otherwise, show that the graph of $y = 3x^2 + 6x + 10$ is always above the x-axis.

Ans: $3x^2 + 6x + 10 = (x+1)^2 + 7$, the minimum value is 7 and the other values are always greater than 7. Hence, always above x-axis.

Ex-10-13: Answer part (i) of this question on the insert provided.

The insert shows the graph of $y = \dfrac{1}{x}$.

(i) On the insert, on the same axes, plot the graph of $y = x^2 - 5x + 5$ for $0 \le x \le 5$.

(ii) Show algebraically that the x-coordinates of the points of intersection of

the curves $y = \dfrac{1}{x}$ and $y = x^2 - 5x + 5$ satisfy the equation $x^3 - 5x^2 + 5x - 1 = 0$.

(iv) Given that $x = 1$ at one of the points of intersection of the curves, factorise $x^3 - 5x^2 + 5x - 1$ into a linear and a quadratic factor.

Show that only one of the three roots of $x^3 - 5x^2 + 5x - 1 = 0$ is

rational.

PAPER-11

SECTION A (C1-14-5-2006)

Ex-11-1: **(i)** Evaluate $\left(\dfrac{1}{27}\right)^{-\frac{2}{3}} = \dfrac{(1)^{-\frac{2}{3}}}{(27)^{-\frac{2}{3}}} = \dfrac{(27)^{\frac{2}{3}}}{(1)^{\frac{2}{3}}} = \dfrac{(27)^{\frac{2}{3}}}{1} = \left(3^3\right)^{\frac{2}{3}} = 3^2 = 9$

(ii) Simplify $\dfrac{\left(4a^2c\right)^4}{64a^5c^7} = \dfrac{4^4 a^8 c^4}{64a^5c^7} = \dfrac{256 a^8 c^4}{64a^5c^7} = \dfrac{4a^3}{c^3}$

Ex-11-2: A is the point (2, - 5) and B is the point (- 6, 3). M is the midpoint of AB. Determine whether the line with equation $y = -3x - 7$ passes through M.

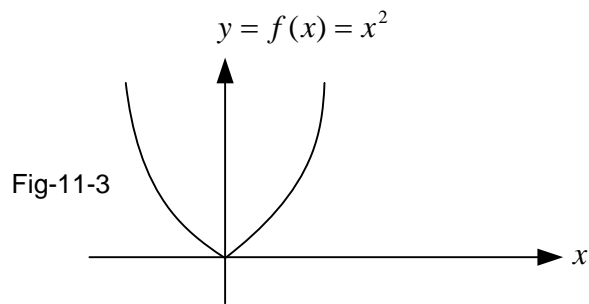$M\left(\dfrac{-6+2}{2}, \dfrac{3-5}{2}\right) = M\left(\dfrac{-4}{2}, \dfrac{-2}{2}\right) = M(-2, -1)$

$y = -3x - 7$

$$-1 \overset{?}{=} -3(-2) - 7$$

$$-1 \overset{?}{=} 6 - 7$$

$$-1 \overset{?}{=} -1 \quad \Rightarrow ok$$

Ex-11-3:   Fig-11-3 shows the graph of $y = f(x)$.

$$y = f(x) = x^2$$

Fig-11-3

Draw the graphs of the following.

   (i) $y = f(x) - 2$

   (ii) $y = f(x-3)$

Ans:

$$y = f(x) = x^2$$

$$y = f(x) = (x-3)^2$$

Fig-11-3

$$y = f(x) = x^2 - 2$$

3,0

0,-2
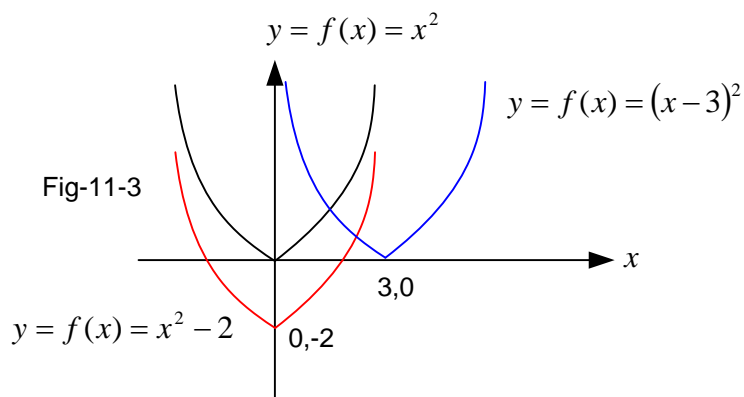
Ex-11-4:   **(i)**   Expand and simplify $\left(3\sqrt{2} - 2\sqrt{3}\right)^2$

Ans: $\left(3\sqrt{2} - 2\sqrt{3}\right)^2 = \left(3\sqrt{2}\right)^2 - 2\left(3\sqrt{2}\right)\left(2\sqrt{3}\right) + \left(2\sqrt{3}\right)^2 = 18 - 12\sqrt{6} + 12 = 30 - 12\sqrt{6}$

(ii) Express $\dfrac{20\sqrt{3}}{\sqrt{30}}$ in the form $a\sqrt{b}$, where a and b are integers and b is as small as possible.

Ans: $\dfrac{20\sqrt{3}}{\sqrt{30}} = \dfrac{20\sqrt{3}\sqrt{30}}{\sqrt{30}\sqrt{30}} = \dfrac{20\sqrt{90}}{30} = \dfrac{20\sqrt{9}\sqrt{10}}{30} = \dfrac{60\sqrt{10}}{30} = 2\sqrt{10}$

Ex-11-5: Make r the subject of $z = \pi r^2(x - y)$, where $r > 0$.

Ans: $z = \pi r^2(x - y)$

$\pi r^2 = \dfrac{z}{(x - y)}$

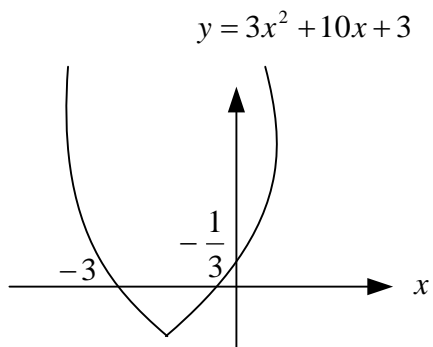$r^2 = \dfrac{z}{\pi(x - y)}$

$r = \pm\sqrt{\dfrac{z}{\pi(x - y)}}$

$r = \sqrt{\dfrac{z}{\pi(x - y)}}$ **OK**

Ex-11-6: Solve the inequality $3x^2 + 10x + 3 > 0$

Ans:

$3x^2 + 10x + 3 > 0$

$(3x + 1)(x + 3) > 0$

$y = 3x^2 + 10x + 3$

(3x + 1) (x + 3) > 0　　(x + 3)　　　　(3x + 1)

　　　　　　　　　　　　　X = - 3　　　　X = - 1/3

-		-		0	+

-	0	+			+

(3x + 1) (x + 3)　　**+**　　　　　**-**　　　　**+**

　　　Ans:　　**- 3 > x**　　**or**　　**x > - 1/3**

Ex-11-7:　　Solve the inequality

(i)　　$\dfrac{4x-5}{7} > 2x+1$

Ans:　$4x-5 > 7(2x+1)$

$4x-5 > 14x+7$

$4x-14x > 7+5$

$-10x > 12$

$x < -\dfrac{12}{10}$

$x < -\dfrac{6}{5}$

(ii)　　$\dfrac{7}{x-3} > 2$

Ans:

$(x-3)^2\,\dfrac{7}{x-3} > 2(x-3)^2$　　　**Very Important.** $(x-3)^2$ is always **positive**
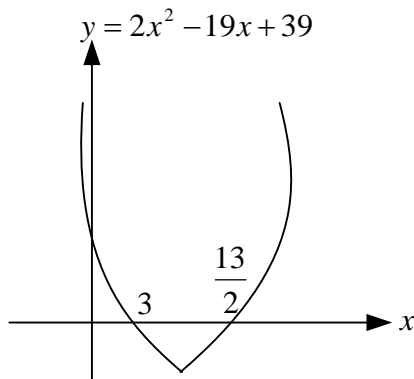
$7(x-3) > 2(x-3)^2$

$7x-21 > 2(x^2-6x+9)$

$2(x^2-6x+9) < 7x-21$

$2x^2 - 12x + 18 < 7x - 21$

$2x^2 - 12x + 18 - 7x + 21 < 0$

$2x^2 - 19x + 39 < 0$

$(2x - 13)(x - 3) < 0$

$y = 2x^2 - 19x + 39$

$3 < x < \dfrac{13}{2}$

Ex-11-8:   Find the coefficient of x^5 in the binomial expansion of $(5 + 2x)^7$.

$$(a+b)^7 = \binom{7}{0}a^7 + \binom{7}{1}a^6 b + \binom{7}{2}a^5 b^2 + \binom{7}{3}a^4 b^3 + \binom{7}{4}a^3 b^4 + \binom{7}{5}a^2 b^5 + \binom{7}{6}ab^6 + \binom{7}{7}b^7$$

looking there properly: $(5 + 2x)^7 = \ldots + \binom{7}{5}5^2(2x)^5 + \ldots$

$$\binom{7}{5} = \frac{7!}{(7-5)!\,5!} = \frac{7 \times 6 \times 5!}{2! \times 5!} = 21 \qquad\qquad \text{and} \qquad 5^2(2x)^5 = 25 \times 2^5 x^5 = 800x^5$$

$$\Rightarrow \binom{7}{5}5^2(2x)^5 = 21 \times 25 \times 32x^5 = 16800\,x^5 \ldots \textbf{ans: } \textbf{16800}$$

Ex-11-9: You are given that $f(x) = 4x^3 + kx + 6$, where k is a constant. When f (x) is divided by $(x-2)$, the remainder is 42. Use the remainder theorem to find the value of k. Hence find a root of $f(x) = 0$.

Ans: $f(x) = 4x^3 + kx + 6$

$f(2) = 4 \times 2^3 + k \times 2 + 6 = 42$

$4 \times 2^3 + k \times 2 + 6 = 42 \Rightarrow 32 + 2k + 6 = 42 \Rightarrow 2k = 42 - 38 = 4 \Rightarrow k = 2$

$f(x) = 4x^3 + 2x + 6$

$f(-1) = 4(-1)^3 + 2(-1) + 6 = -6 + 6 = 0 \Rightarrow x = -1$ is a root

Ex-11-10: You are given that n, $n + 1$ and $n + 2$ are three consecutive integers.

(i) Expand and simplify $n^2 + (n+1)^2 + (n+2)^2$

$n^2 + (n+1)^2 + (n+2)^2 = n^2 + n^2 + 2n + 1 + n^2 + 4n + 4 = 3n^2 + 6n + 5$

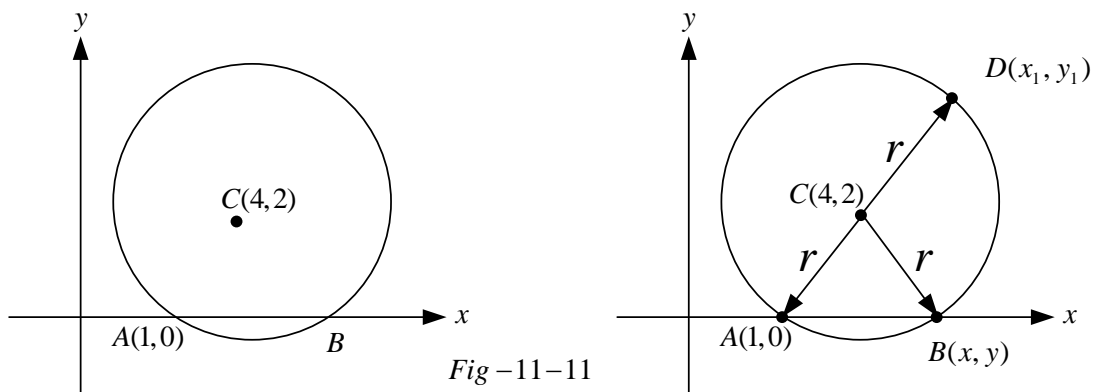**(ii)** For what values of n will the sum of the squares of these three consecutive integers be an even number?

$n^2 + (n+1)^2 + (n+2)^2 = 3n^2 + 6n + 5 = 3n(n+2) + 5$

It is even for $n = 1, 3, 5, ... n = odd$

Give a reason for your answer.

SECTION B (P-11)

Ex-11-11: Fig-11-11 shows a sketch of a circle with centre C (4, 2). The circle intersects the x-axis at A(1, 0) and at B.

Fig −11−11

(i) Write down the coordinates of B.

Ans:

$$r = \sqrt{(4-x)^2 + (2-0)^2} = \sqrt{13}$$

$$r^2 = (4-x)^2 + (2-0)^2 = 13$$

$$(4-x)^2 = 13 - 4 = 9$$

$$(4-x) = \pm 3$$

(a) $\quad 4 - x = 3 \Rightarrow -x = -1 \Rightarrow x = 1$

(b) $\quad 4 - x = -3 \Rightarrow -x = -7 \Rightarrow x = 7$

$$\Rightarrow B(7,0)$$

(ii) Find the radius of the circle and hence write down the equation of the circle.
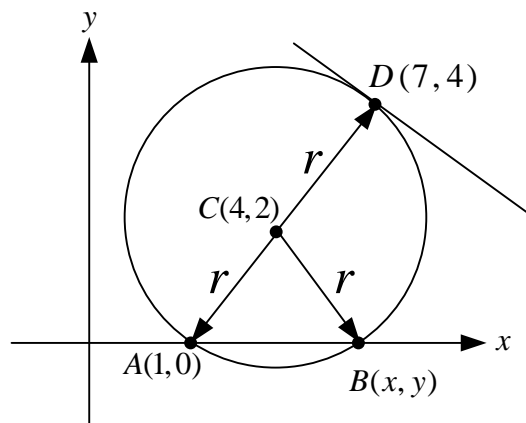
Ans: **Already found**

(iii) AD is a diameter of the circle. Find the coordinates of D.

Ans: $\quad 4 = \dfrac{x_1 + 1}{2} \Rightarrow x_1 = 8 - 1 = 7$

$$2 = \dfrac{y_1 + 0}{2} \Rightarrow y_1 = 4 \qquad\qquad \Rightarrow D(7,4)$$

(iv) Find the equation of the tangent to the circle at D. Give your answer in the form $y = ax + b$.
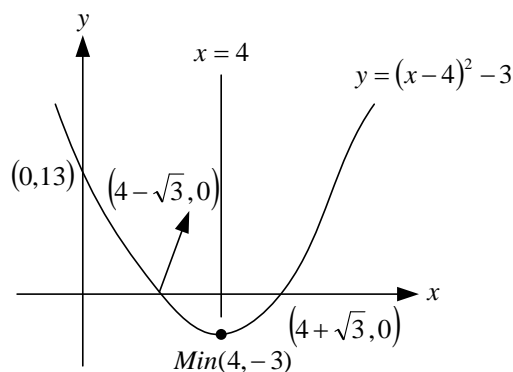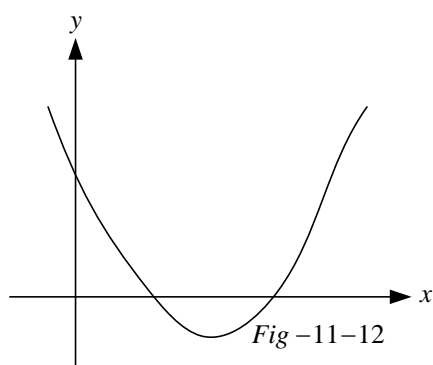
Ans:

$$\text{Gradient of } CD = m = \frac{4-2}{7-4} = \frac{2}{3} \Rightarrow m_2 = -\frac{1}{m_1} = -\frac{3}{2}$$

$$CD = m = \frac{4-2}{7-4} = \frac{2}{3} \Rightarrow m_2 = -\frac{1}{m_1} = -\frac{3}{2}$$

$$4 = -\frac{3}{2}(7) + c \Rightarrow c = 4 + \frac{21}{2} = \frac{29}{2}$$

$$y = -\frac{3}{2}x + \frac{29}{2}$$

Ex-11-12: Fig-11-12 shows a sketch of the curve with equation $y = (x-4)^2 - 3$

$Fig-11-12$

(i) Write down the equation of the line of symmetry of the curve and the coordinates of the minimum point. See Fig.12

(ii) Find the coordinates of the points of intersection of the curve with the *x*-axis and the *y*-axis, using surds where necessary. See Fig.12

(iii) The curve is translated by $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Show that the equation of the translated curve may be written as $y = x^2 - 12x + 33$

Ans: $y = f(x) = (x-4)^2 - 3 = x^2 - 8x + 16 - 3 = x^2 - 8x + 13$

$f(x-2) = (x-2)^2 - 8(x-2) + 13 = x^2 - 4x + 4 - 8x + 16 + 13$

$f(x-2) = x^2 - 12x + 33$

(iv) Show that the line $y = 8 - 2x$ meets the curve $y = x^2 - 12x + 33$ at just one point, and find the coordinates of this point.
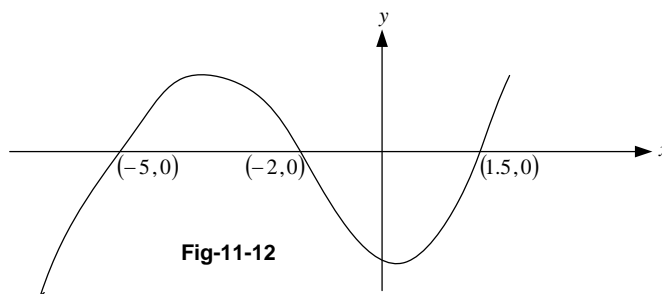
Ans: $y = 8 - 2x$

$$y = x^2 - 12x + 33$$

$$\Rightarrow x^2 - 12x + 33 = 8 - 2x \Rightarrow x^2 - 12x + 33 - 8 + 2x = 0 \Rightarrow x^2 - 10x + 25 = 0$$

$$\Rightarrow (x - 5)^2 = 0 \Rightarrow x_1 = x_2 = 5 \quad \Rightarrow P(5, -2) \text{ only point.}$$

Ex-11-12: Fig-11-12 shows the graph of a cubic curve. It intersects the axes at (-5, 0), (-2, 0), (1.5, 0) and (0, -30).

Fig-11-12

(i) Use the intersections with both axes to express the equation of the curve in a factorised form.

Ans:

$$(x + 5)(x + 2)(x - 1.5) = 0$$

(ii) Hence show that the equation of the curve may be written as
$y = 2x^3 + 11x^2 - x - 30$

(iii) Draw the line $y = 5x + 10$ accurately on the graph. The curve and this line intersect at (-2, 0); find graphically the x-coordinates of the other points of intersection.

(iv) Show algebraically that the x-coordinates of the other points of intersection satisfy the equation $2x^2 + 7x - 20 = 0$.

Hence find the exact values of the x-coordinates of the other points of intersection

PAPER-12

Ex-12-1: Differentiate $x + \sqrt[5]{x^3}$

Ans: $Let \quad y = x + \sqrt[5]{x^3} = x + x^{\frac{3}{5}} \quad \Rightarrow \frac{dy}{dx} = 1 + \frac{3}{5}x^{-\frac{2}{5}} = 1 + \frac{3}{5\sqrt[5]{x^2}}$

Ex-12-2: The nth term of an arithmetic progression is $3 + 2n$. Find the sum of the first 30 terms.

Ans: $a_n = 3 + 2n$

$a_n = 3 + 2n \Rightarrow a_1 = 3 + 2 = 5$

$a_1 = 3 + 2 = 5$

$a_2 = 3 + 2 \times 2 = 7$

$a_3 = 3 + 2 \times 3 = 9$

$d = a_2 - a_1 = 7 - 5 = 2$

$S_n = \frac{n}{2}\left[2a_1 + (n-1)d\right] = \frac{30}{2}\left[2 \times 5 + 2(30 - 1)\right] = 1020$

Ex-12-3: Given that $\sin\theta = \frac{\sqrt{3}}{2}$, find in surd form the possible values of $\cos\theta$.

Ans:
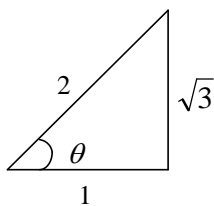
Or $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{3}{4}} = \frac{\sqrt{4-3}}{2} = \frac{1}{2}$

Ex-12-4: A curve has equation $y = x + \frac{1}{x}$

Use calculus to show that the curve has a turning point at x = 1.

$\dfrac{dy}{dx} = slope = gradient = 0$ **and** $\dfrac{d^2y}{dx^2} < 0$

At this point (i.e. Maximum or turning point)

At this point (i.e. Minimum or turning point)

$\dfrac{dy}{dx} = slope = gradient = 0$ **and** $\dfrac{d^2y}{dx^2} > 0$

Ans: $y = x + \dfrac{1}{x} = x + x^{-1}$ $\Rightarrow \dfrac{dy}{dx} = 1 - x^{-2} = 1 - \dfrac{1}{x^2} = \dfrac{x^2 - 1}{x^2} = 0$ $\Rightarrow x = \pm 1$
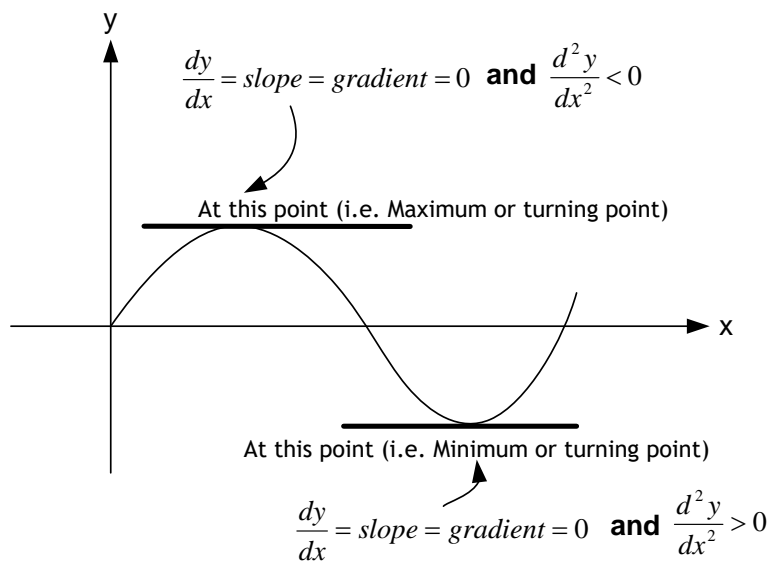
Show also that this point is a minimum.

For minimum

$\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} > 0$ $\Rightarrow \dfrac{d^2y}{dx^2} = 2x^{-3} = \dfrac{2}{x^3}\bigg|_{x=1} = 2 > 0$ $\Rightarrow Minimum$

Ex-12-5:   (i)   Write down the value of $\log_5 25$

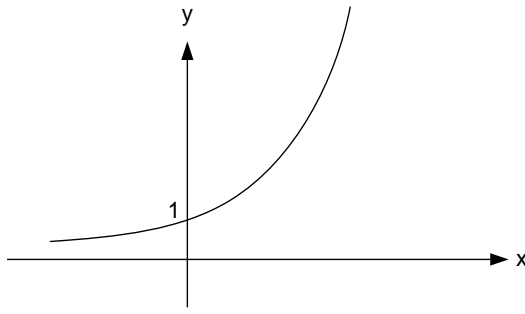   **Ans:** $\log_5 25 = \log_5 5^2 = 2\log_5 5 = 2$

    (ii)   Find $\log_3\left(\dfrac{1}{9}\right)$

    Ans: $\log_3\left(\dfrac{1}{9}\right) = \log_3(3)^{-2} = -2\log_3 3 = -2$

    (iii)   Express $\log_a x + \log_a\left(x^5\right)$ as a multiple of $\log_a x$

    Ans: $\log_a x + \log_a\left(x^5\right) = \log_a x + 5\log_a x = 6\log_a x$

Ex-12-6:   Sketch the graph $y = 3^x$

Solve the equation $3^x = 50$, giving your answer correct to 2 decimal places.

Ans:

$$3^x = 50 \Rightarrow \log 3^x = \log 50 \Rightarrow x \log 3 = \log 50 \Rightarrow x = \frac{\log 50}{\log 3} = \frac{1.699}{0.477} = 3.561 \cong 3.56$$

Ex-12-7: The gradient of a curve is given by $\dfrac{dy}{dx} = \dfrac{6}{x^3}$. The curve passes through (1, 4).

Find the equation of the curve.

Ans:

$$\frac{dy}{dx} = \frac{6}{x^3} \Rightarrow \frac{dy}{dx} = 6x^{-3} \Rightarrow dy = 6x^{-3} dx \Rightarrow \int dy = \int 6x^{-3} dx \Rightarrow y = \frac{6x^{-3+1}}{-2} + c$$

$$y = \frac{6x^{-3+1}}{-2} + c = -\frac{3}{x^2} + c \Rightarrow 4 = -\frac{3}{1} + c \Rightarrow c = 7 \Rightarrow y = -\frac{3}{x^2} + 7$$

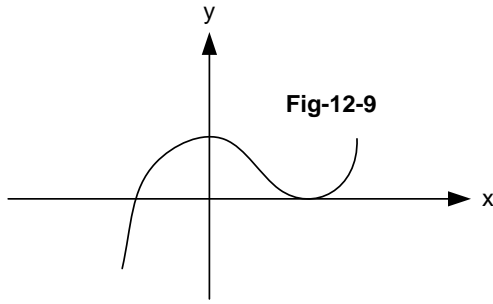**Ex-12-8:** (i) Solve the equation $\cos x = 0.4$ for $0^0 \le x \le 360^0$

Ans: $\cos x = 0.4 \Rightarrow x = 66.422^0, \ 293.58^0$

(ii) Describe the transformation which maps the graph of $y = \cos x$ onto the graph of $y = \cos 2x$

Ans: The values of x will be stretched by half and y will remain the same.

SECTION B (P-12)

Ex-12-9: Fig-12-9 shows a sketch of the graph of $y = x^3 - 10x^2 + 12x + 72$

Fig-12-9

(i) Write down $\dfrac{dy}{dx}$.

Ans: $y = x^3 - 10x^2 + 12x + 72 \qquad \Rightarrow \dfrac{dy}{dx} = 3x^2 - 20x + 12$

(ii) Find the equation of the tangent to the curve at the point on the curve where x=2

Ans: gradient of the tangent at x=2

$\dfrac{dy}{dx} = 3x^2 - 20x + 12 \Big|_{x=2} = 12 - 40 + 12 = -16$

This line is tangent at point (2, 64) on the curve.

The equation is: $\qquad y = mx + c = -16x + c \qquad \Rightarrow 64 = -16(2) + c \quad \Rightarrow c = 96$

The equation is: $\qquad y = -16x + 96$

(iii) Show that the curve crosses the x-axis at x = -2. Show also that the curve touches the x-axis at x = 6.

If the curve crosses the x-axis at x=-2, then $y\Big|_{x=-2} = 0$

$\Rightarrow y\Big|_{x=-2} = x^3 - 10x^2 + 12x + 72 \Big|_{x=-2} = -8 - 40 - 24 + 72 = 0 \quad OK$

The equation is: $\qquad y = mx + c = -16x + c \qquad \Rightarrow 64 = -16(2) + c \quad \Rightarrow c = 96$

(iv) Find the area of the finite region bounded by the curve and the x-axis, shown shaded in Fig.9

Ans: $Area = \displaystyle\int_{-2}^{6} \left(x^3 - 10x^2 + 12x + 72\right)dx = \dfrac{x^4}{4} - \dfrac{10x^3}{3} + \dfrac{12x^2}{2} + 72x \Big|_{-2}^{6}$

$\Rightarrow Area = \dfrac{(6)^4}{4} - \dfrac{10(6)^3}{3} + \dfrac{12(6)^2}{2} + 72(6) - \left(\dfrac{(-2)^4}{4} - \dfrac{10(-2)^3}{3} + \dfrac{12(-2)^2}{2} + 72(-2)\right)$

$$\Rightarrow Area = 324 - 720 + 216 + 432 - (4 + 26.667 + 24 - 144) = 252 + 89.333 = 341.333$$

Ex-12-10:   Arrowline Enterprises is considering two possible logos:

and
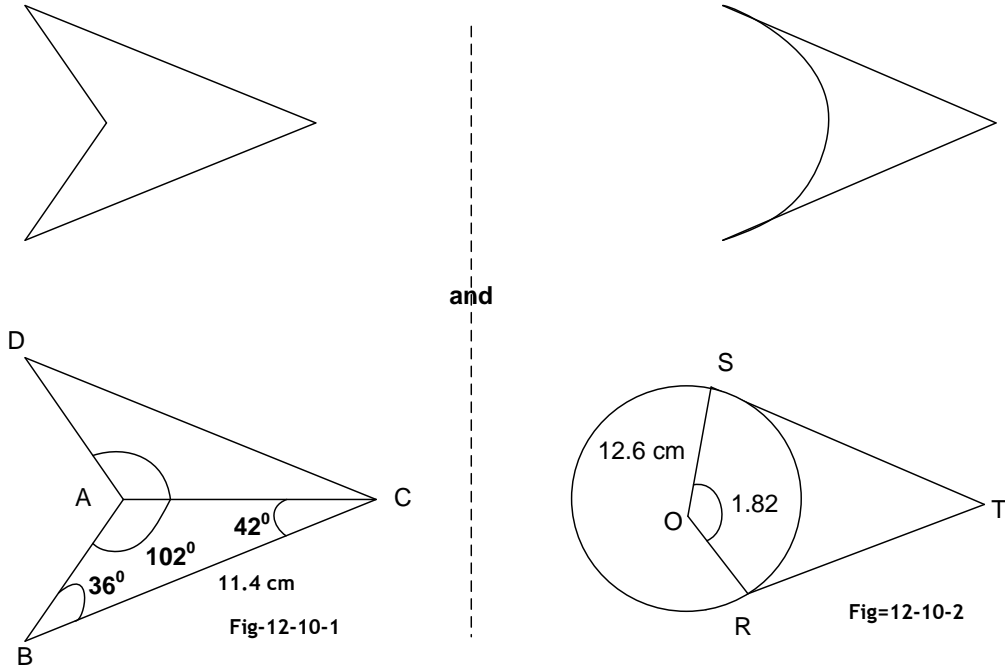
Fig-12-10-1

Fig=12-10-2

(i)   Fig-12-10-1 shows the first logo ABCD. It is symmetrical about AC.

Find the length of AB and hence find the area of this logo.

$$\frac{11.4}{\sin 102^0} = \frac{AB}{\sin 42^0} \qquad \Rightarrow AB = \frac{11.4 \times \sin(42^0)}{\sin 102^0} = 7.8 \ cm$$

$$Area = \frac{1}{2}\left(11.4 \times 7.8 \times \sin(36^0)\right) = 26.133 \ cm^2$$

(ii)   Fig-12-2 shows a circle with centre O and radius 12.6 cm. ST and RT are tangents to the circle and angle SOR is 1.82 radians. The shaded region shows the second logo.

Fig-12-10-2
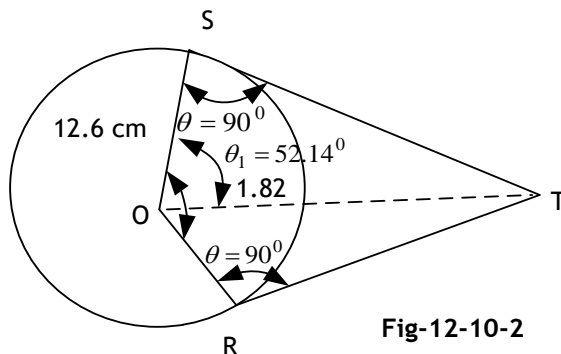
Show that ST=16.2 cm to 3 significant figures.

Ans: $182\ radian = \dfrac{1.82(180)}{\pi} = 104.28^{0}$ $\Rightarrow \theta_1 = \dfrac{104.28^{0}}{2} = 52.14^{0}$

$\tan\theta_1 = \tan 52.14^{0} = \dfrac{ST}{12.6}$ $\Rightarrow ST = 12.6\tan 52.14^{0} = 16.21\ cm$

Find the area and perimeter of this logo.

Ans:

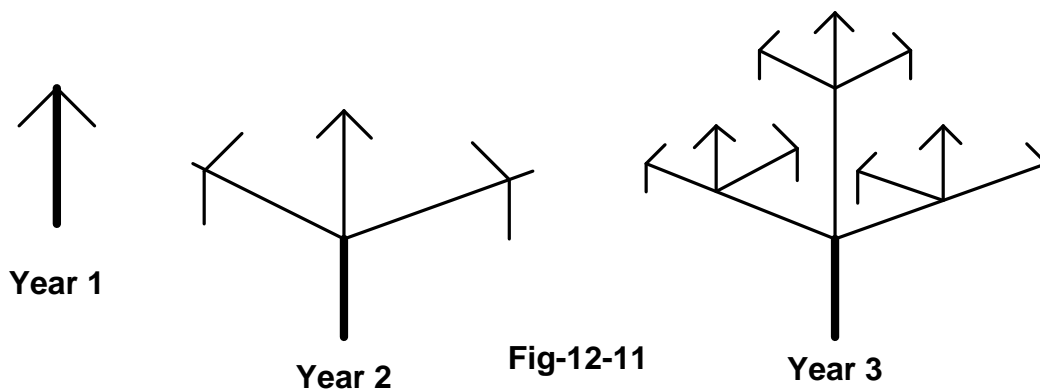$Area = \left(\dfrac{12.6\times 16.21}{2}\right) + \left(\dfrac{12.6\times 16.21}{2}\right) - \dfrac{1}{2}(12.6)^2(1.82) = 204.246 - 144.472 = 59.774\ cm^2$

Ans:

$SR = r\theta = 12.6\times 1.82 = 22.932\ cm \Rightarrow Perimeter = ST + RT + SR = 16.21 + 16.21 + 22.932 = 55.352\ cm$

Ex-12-11: There is a flowerhead at the end of each stem of an oleanderplant. The next year, each flowerhead is replaced by three stems and flowerheads, as shown in **Fig-12-11**.

Year 1

Year 2   **Fig-12-11**   **Year 3**

(i)   How many flowerheads are there in year 5?

Ans: Flowerheads are: 1, 3, 9,   therefore, geometric with ratio=r=3

$a_n = a_1 r^{n-1} \Rightarrow a_5 = a_1 r^{5-1} = (3)^4 = 81$

Find also the sum to infinity of this progression.

Ans: $S = \dfrac{a_1}{1-r}\ for\ \inf inite\ GP \Rightarrow S = \dfrac{a_1}{1-r} = \dfrac{4}{1-0.5} = \dfrac{4}{0.5} = 8$

(ii)   How many flowerheads are there in year n?

Ans: $a_n = a_1 r^{n-1} = (3)^{n-1}$

(iii)   As shown in Fig.11, the total number of stems in year 2 is 4, (that is, 1 old one and 3 new ones). Similarly, the total number of stems in year 3 is 13, (that is, 1+3+9).

Show that the total number of stems in year n is given by $\dfrac{3^n - 1}{2}$

Ans: $S = \dfrac{a_1}{1-r}$ for $\inf inite$ GP $\Rightarrow S = \dfrac{a_1}{1-r} = \dfrac{4}{1-0.5} = \dfrac{4}{0.5} = 8$

     (iv)   Kitty's oleander has a total of 364 stems. Find

         (A)   Its age,

$a_n = \dfrac{3^n - 1}{2} = 364$ $\Rightarrow 3^n - 1 = 728$ $\Rightarrow 3^n = 729$ $\Rightarrow n \log 3 = \log 729$

$\Rightarrow n = \dfrac{\log 729}{\log 3} = \dfrac{2.863}{0.477} = 6$ $years$

         (B)   How many flowerheads it has.

$a_n = a_1 r^{n-1} = (3)^{n-1} = 3^{6-1} = 3^5 = 243$

     (v)   Abdul's oleander has over 900 flowerheads.

Show that its age, y years, satisfies the inequality $y > \dfrac{\log_{10} 900}{\log_{10} 3} + 1$

$a_n = (3)^{y-1} > 900$

$(y - 1) \log 3 > \log 900$ $\Rightarrow y - 1 > \dfrac{\log_{10} 900}{\log_{10} 3}$ $\Rightarrow y > \dfrac{\log_{10} 900}{\log_{10} 3} + 1$

Find the smallest integer value of y for which this is true.

$y > \dfrac{\log_{10} 900}{\log_{10} 3} + 1 > 6.19 + 1 > 7.19$ $y = 8$ $years$

PAPER-13

Ex-13-1: Differentiate $6x^{\frac{5}{2}} + 4$

$$y = 6x^{\frac{5}{2}} + 4$$

$$\frac{dy}{dx} = 6 \times \frac{5}{2} x^{\frac{3}{2}} = 15x^{\frac{3}{2}}$$

Ex-13-2: A geometric progression has 6 as its first term. Its sum to infinity is 5. Calculate its common ratio.

$$a_n = a_1 r^{n-1}$$

$$s_{\to\infty} = \frac{a_1}{1-r}$$

$$s_{\to\infty} = \frac{a_1}{1-r} = \frac{6}{1-r} = 5 \quad \Rightarrow 5 - 5r = 6 \quad \Rightarrow -5r = 6 - 5 = 1 \quad \Rightarrow r = -\frac{1}{5}$$

Ex-13-3: Given that $\cos\theta = \frac{1}{3}$ and θ is acute, find the exact value of $\tan\theta$.

$$\cos\theta = \frac{1}{3}$$
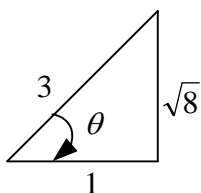
$$\sqrt{9-1} = \sqrt{8}$$

$$\tan\theta = \frac{\sqrt{8}}{1} = \sqrt{8}$$

Ex-13-4: Sequences A, B and C are shown below. They each continue in the pattern established by the given terms.

A: 1,   2,   4,   8,   16,   32,   ...

B: 20,   -10,   5,   -2.5,   1.25,   -0.625,   ...

C: 20,   5,   1,   20,   5,   1,   ...

(i) Which of these sequences is periodic? **Ans:** C is periodic

(ii) Which of these sequences is convergent? **Ans:** B is convergent

(iii) Find, in terms of n, the nth term of sequence A.

Ans: $a_n = a_1 r^{n-1} = 2^{n-1}$

Ex-13-5: A is the point (2, 1) on the curve $y = \dfrac{4}{x^2}$.

B is the point on the same curve with x-coordinate 2.1.

(i) Calculate the gradient of the chord AB of the curve. Give your

answer correct to 2 decimal places.

$$y = \frac{4}{x^2}$$

$$y = \frac{4}{x^2}\bigg|_{x=2.1} = \frac{4}{(2.1)^2} = \frac{4}{4.41} = 0.91$$

A(2, 1) and B(2.1, 0.91)

$$m = gradient = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0.91 - 1}{2.1 - 2} = -\frac{0.09}{0.1} = -0.9$$

(ii) Give the x-coordinate of a point C on the curve for which the gradient of chord AC is a better approximation to the gradient of the curve at A.

(iii) Use calculus to find the gradient of the curve at A.

$$y = \frac{4}{x^2} = 4x^{-2}$$

$$\frac{dy}{dx} = -8x^{-3} = -\frac{8}{x^3}\bigg|_{x=2} = -\frac{8}{8} = -1$$

Ex-13-6: Sketch the curve $y = \sin x$ for $0^0 \leq x \leq 360^0$
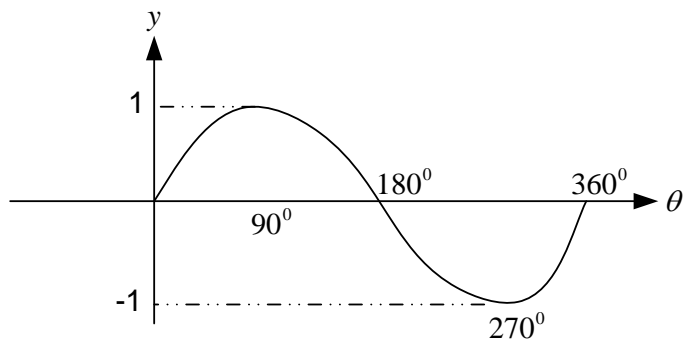
Solve the equation $\sin x = -0.68$ for $0^0 \leq x \leq 360^0$

$$\theta = 180 + 42.84^0 = 222.84^0 \quad and \quad \theta = 360 - 42.84^0 = 317.16^0$$
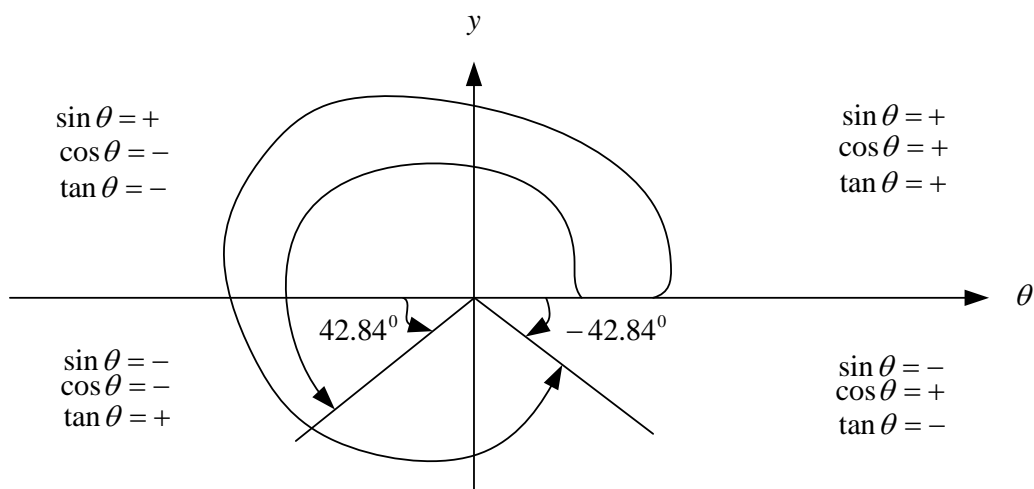
Ex-13-7: The gradient of a curve is given by $\dfrac{dy}{dx} = x^2 - 6x$. Find the set of values of x for which y is an increasing function of x.

$$\frac{dy}{dx} = x^2 - 6x$$

$$\frac{dy}{dx} = x^2 - 6x > 0 \qquad x(x - 6) > 0$$

$0 > x$ OR $x > 6$

Ex-13-8: The 7th term of an arithmetic progression is 6. The sum of the first 10 terms of the progression is 30.

Find the 5th term of the progression.

$$a_n = a_1 + (n-1)d$$

$$6 = a_1 + (7-1)d \quad \Rightarrow a_1 + 6d = 6 \qquad ....(1)$$

$$s = \frac{n}{2}\left[2a_1 + (n-1)d\right] \quad \Rightarrow 30 = \frac{10}{2}(2a_1 + 9d) \quad \Rightarrow 10a_1 + 45d = 30 \qquad ....(2)$$

$$\Rightarrow 2a_1 + 9d = 6 \qquad ....(2)$$

$$\Rightarrow a_1 + 6d = 6 \qquad ....(1)$$

$$d = 2 \quad and \quad a_1 = -6$$

$$a_n = a_1 + (n-1)d \Rightarrow a_5 = -6 + (5-1)(2) = -6 + 8 = 2$$

Ex-13-9: A curve has gradient given by $\dfrac{dy}{dx} = 6x^2 + 8x$. The curve passes through the point (1, 5). Find the equation of the curve.

$$\frac{dy}{dx} = 6x^2 + 8x$$

$$dy = \left(6x^2 + 8x\right)dx$$

$$y = \int\left(6x^2 + 8x\right)dx = \frac{6x^3}{3} + \frac{8x^2}{2} = 2x^3 + 4x^2 + c$$

$$5 = 2(1)^3 + 4(1)^2 + c \quad \Rightarrow c = 5 - 2 - 4 = -1$$

$$y = 2x^3 + 4x^2 - 1$$

Ex-13-10: (i) Express $\log_a x^4 + \log_a\left(\dfrac{1}{x}\right)$ as a multiple of $\log_a x$

$$\log_a x^4 + \log_a\left(\frac{1}{x}\right) = 4\log_a x + \log_a x^{-1} = 4\log_a x - \log_a x = 3\log_a x$$

(ii) Given that $\log_{10} b + \log_{10} c = 3$, find b in terms of c.

$$\log_{10} b + \log_{10} c = 3$$

$$\log_{10} bc = 3$$

$$bc = 10^3 = 1000 \qquad \Rightarrow b = \frac{1000}{c}$$

Ex-13-11: **Fig-13-11a** shows a village green which is bordered by 3 straight roads AB, BC and CA. The road AC runs due North and the measurements shown are in meters.

Fig-13-11a
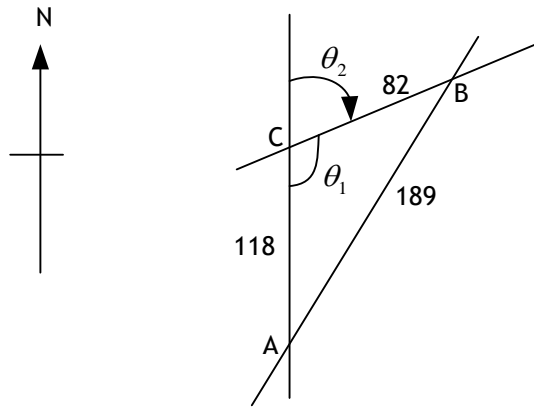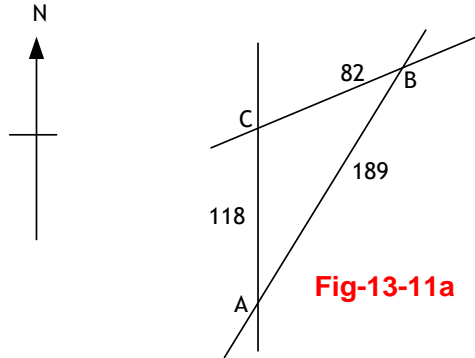
(i) Calculate the bearing of B from C, giving your answer to the nearest 0.1^0

$$(189)^2 = (118)^2 + (82)^2 - 2(118)(82)\cos\theta_1 \quad \Rightarrow \cos\theta_1 = \frac{(118)^2 + (82)^2 - (189)^2}{2(118)(82)} = -0.7789$$

$$\Rightarrow \cos\theta_1 = -0.7789 \quad \Rightarrow \theta_1 = 141.16^0$$

$$\Rightarrow \cos\theta_1 = -0.7789 \quad \Rightarrow \theta_1 = 141.16^0$$

Ans: bearing of B from C: $\theta_2 = 180^0 - 141.16^0 = 38.84^0 \cong 38.8^0$

(ii) Calculate the area of the village green

$$A(village-green) = \frac{1}{2}(118)(82)\sin\theta_1 = 3031.5$$

The road AB is replaced by a new road, as shown in **Fig-13-11b**. The village green is extended up to the new road.

Fig-13-11b
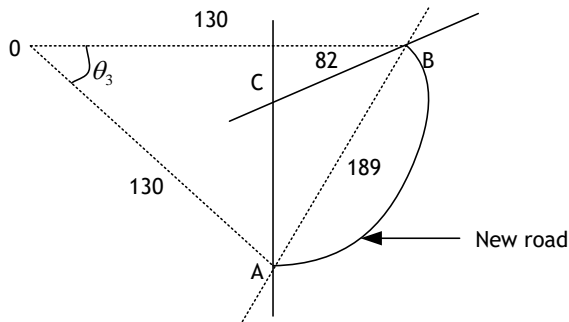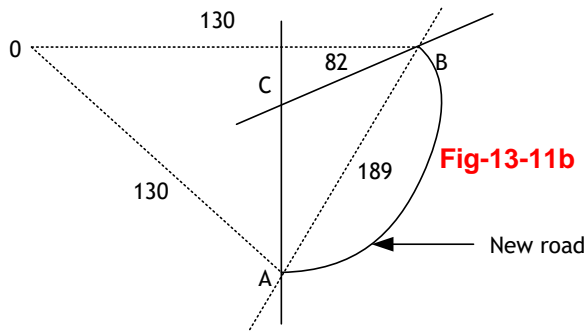
$$(189)^2 = (130)^2 + (130)^2 - 2(130)(130)\cos\theta_3 \quad \Rightarrow \cos\theta_3 = \frac{(130)^2 + (130)^2 - (189)^2}{2(130)(130)} = -0.0568$$

$$\Rightarrow \cos\theta_3 = -0.0568 \Rightarrow \theta_3 = 93.256^0 = \frac{93.256^0 \times \pi}{180} = 1.6276 \cong 1.63$$

The new road is an arc of a circle with centre O and radius 130 m.

(iii)   (A)   Show that angle AOB is 1.63 radians, correct to 3 significant figures.

       (B)   Show that the area of land added to the village green is $5300 \ m^2$ correct to 2 significant figures.

$$Land \ added = \frac{1}{2}(130)^2(1.63) - \frac{1}{2}(130)(130)\sin 93.256^0 = 13773 - 8438 = 53355$$

Ex-13-12:   Fig-13-12 is a sketch of the curve $y = 2x^2 - 11x + 12$

Fig-13-12
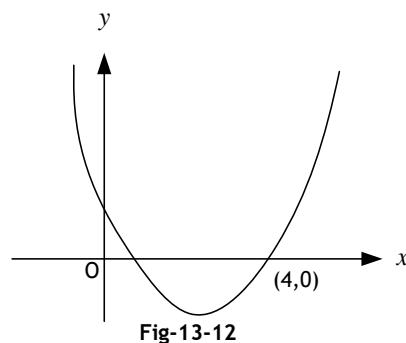
(i) Show that the curve intersects the x-axis at (4, 0) and find the coordinates of the other point of intersection of the curve and the x-axis.

$$y = 2x^2 - 11x + 12$$

$$y = 2x^2 - 11x + 12 = 0 \quad \Rightarrow (2x - 3)(x - 4) = 0 \quad \Rightarrow x = 4 \quad and \quad x = \frac{3}{2}$$

Point on the x-axis: (4,0)

(ii) Find the equation of the normal to the curve at the point (4,0).

Show also the area of the triangle bounded by this normal and the axes is $1.6 \ unit^2$

$$y = 2x^2 - 11x + 12$$

$$\left.\frac{dy}{dx}\right|_{x=4} = 4x - 11 \bigg|_{x=4} = 16 - 11 = 5$$

$$m_2 = -\frac{1}{m_1} = -\frac{1}{5} \quad y = mx + c = -\frac{1}{5}x + c \quad \Rightarrow 0 = -\frac{1}{5}(4) + c = 0 \quad \Rightarrow c = \frac{4}{5}$$

$$y = -\frac{1}{5}x + \frac{4}{5}$$

(iii) Find the area of the region bounded by the curve and the x-axis.

$$A = \frac{1}{2}bh = \frac{1}{2}(4)\left(\frac{4}{5}\right) = \frac{16}{10} = 1.6 \ m$$

$\left(0, \frac{4}{5}\right)$

$y = -\frac{1}{5}x + \frac{4}{5}$

O   (4,0)
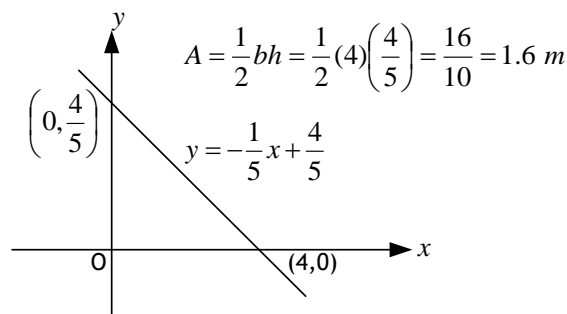
Ex-13-13: Answer part (ii) of this question on the insert provided.

The table gives a firm's monthly profits for the first few months after the start of its business, rounded to the nearest £100 .

Number of months after start-up (x)	1	2	3	4	5	6
Profit for this month (£ y)	500	800	1200	1900	3000	4800

The firm's profits, £ y , for the xth month after start-up are modeled by

$y = k \times 10^{ax}$, where a and k are constants.

(i) Show that , according to this model, a graph of $\log_{10} y$ against x gives a straight line of gradient a and intercept $\log_{10} k$.

$$y = k \times 10^{ax}$$

$$\log y = \log(k \times 10^{ax}) = \log k + ax \log 10 = ax + \log k$$

$$\log y = ax + \log k$$

$$y = mx + c$$

(ii) On the insert, complete the table and plot $\log_{10} y$ against x, drawing by eye a line of best fit.

(iii) Use your graph to find an equation for y in terms of x for this model.

(iv) For which month after start-up does this model predict profits of about £75000 ?

(v) State one way in which this model is unrealistic.

Number of months after start-up (x)	1	2	3	4	5	6
Profit for this month (£ y)	500	800	1200	1900	3000	4800
$\log_{10} y$	2.699	2.9	3.079	3.279	3.477	3.681
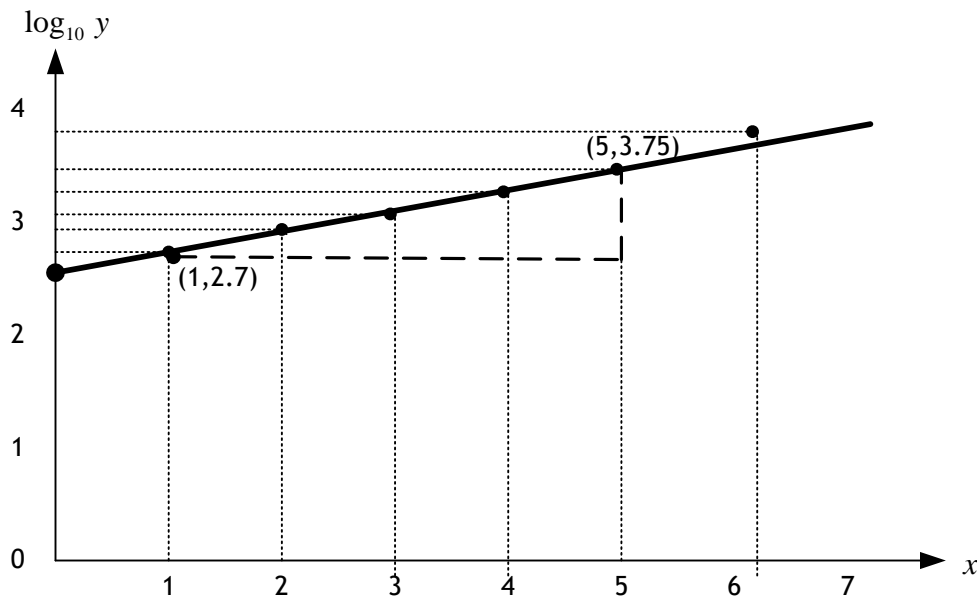
$$\log y = ax + \log k$$

$$\log k = c \cong 2.5 \implies k = 316.22$$

$$a = gradient = m = \frac{3.75 - 2.7}{5 - 1} = \frac{1.05}{4} = 0.263$$

$$y = k \times 10^{ax} = 316.22 \times 10^{0.263t}$$

$$y = 316.22 \times 10^{0.263t}$$

$$75000 = 316.22 \times 10^{0.263t} \quad \Rightarrow 10^{0.263t} = \frac{75000}{316.22} = 237.2 \quad \Rightarrow 0.263t = \log(237.2) = 2.375$$

$$\Rightarrow t = \frac{2.375}{0.263} = 9 \ months$$

Number of months after start-up (x)	1	2	3	4	5	6
Profit for this month (£y)	500	800	1200	1900	3000	4800
$\log_{10} y$	2.70	2.9	3.079	3.279	3.477	3.681

$$\log y = ax + \log k$$

$$\log k = c \cong 2.5 \quad \Rightarrow k = 316.22$$

$$a = gradient = m = \frac{3.681 - 2.7}{6 - 1} = \frac{0.981}{5} = 0.1962$$

$$y = k \times 10^{ax} = 316.22 \times 10^{0.1962t}$$
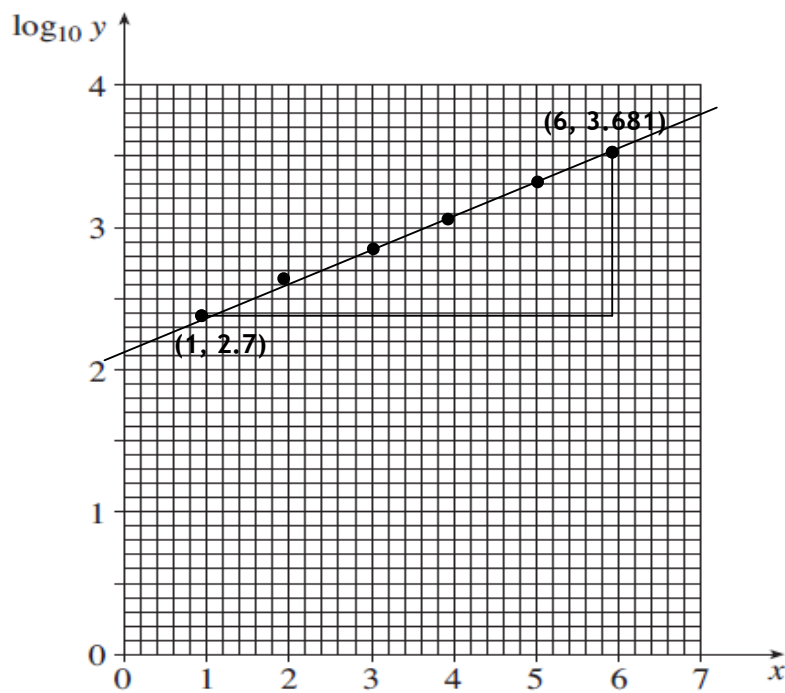
$$y = 316.22 \times 10^{0.1962t}$$

$$75000 = 316.22 \times 10^{0.1962t} \implies 10^{0.1962t} = \frac{75000}{316.22} = 237.2 \implies 0.1962t = \log(237.2) = 2.375$$

$$\implies t = \frac{2.375}{0.1962} = 12.1 \; months$$

PAPER-14

SECTION A(C2-9-1-08)

Ex-14-1: Differentiate $10x^4 + 12$

Ans: $let \;\; y = 10x^4 + 12 \implies \dfrac{dy}{dx} = 40x^3$

Ex-14-2: A sequence begins

1   2   3   4   5   1   2   3   4   5   1   ...

And continues in this pattern

(i) Find the 48th term of this sequence.

Ans: 48/5=9 and 3 remainder. Hence, 3 is the 48th term.

(ii) Find the sum of the first 48 terms of this sequence

Ans: 1+2+3+4+5=15. Hence Sum=9x15+1+2+3=135+6=141

Ex-14-3 You are given that $\tan\theta = \dfrac{1}{2}$ and the angle θ is acute. Show, without using a calculator, that $\cos^2\theta = \dfrac{4}{5}$.

Ans:
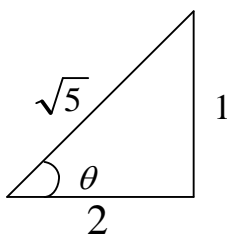
$$\cos\theta = \frac{2}{\sqrt{5}} \implies \cos^2\theta = \left(\frac{2}{\sqrt{5}}\right)^2 = \frac{4}{5}$$

Ex-14-4: Fig-14-4 shows a sketch of the graph $y = f(x)$. On separate diagrams, sketch the graphs of the following, showing clearly the coordinates of the points corresponding to A, B and C.

Fig-14-4

y=f(x)
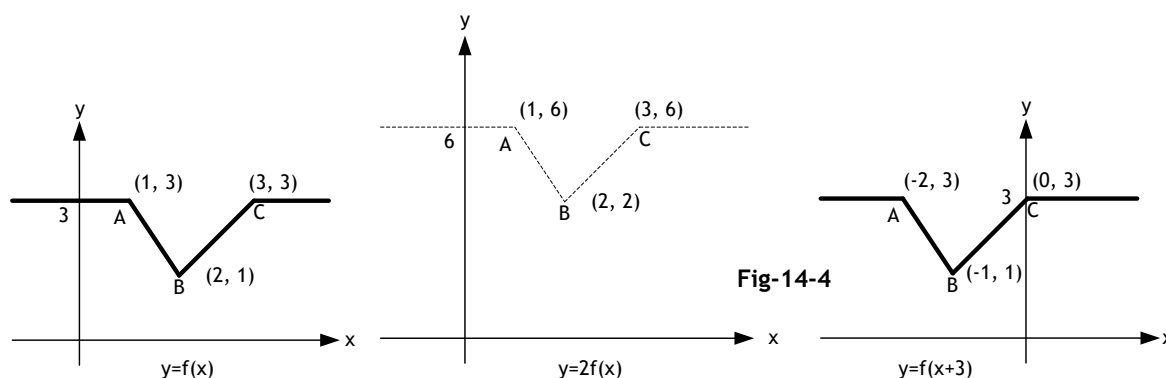
y=2f(x)

y=f(x+3)

(i) $y = 2f(x)$

(ii) $y = f(x+3)$

Ex-14-5:

$$\int \left(12x^5 + \sqrt[3]{x} + 7\right)dx = \int \left(12x^5 + x^{\frac{1}{3}} + 7\right)dx = \frac{12x^6}{6} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + 7x + C = 2x^6 + \frac{3}{4}x^{\frac{4}{3}} + 7x + C$$

Ex-14-6: (i)   Sketch the graph of $y = \sin\theta$ for $0 \le \theta \le 2\pi$
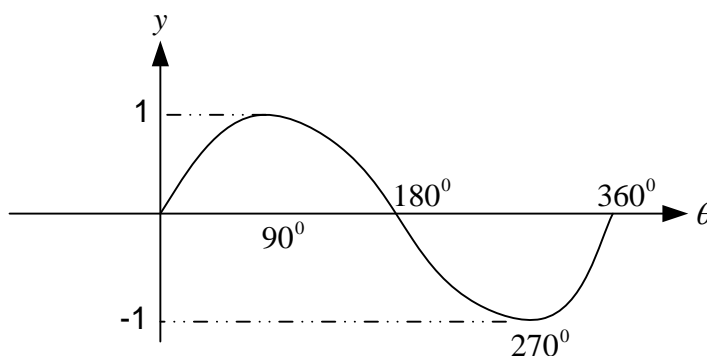
Ans:

(ii)   Solve the equation $2\sin\theta = -1$ for $0 \le \theta \le 2\pi$. Give your answer in the form $k\pi$

Ans:

$$2\sin\theta = -1 \qquad \Rightarrow \sin\theta = -\frac{1}{2} \qquad \Rightarrow \theta = \sin^{-1}(0.5) = 210^0,\ 330^0$$

$$\Rightarrow \theta = \frac{210(\pi)}{180} = \frac{7}{6}\pi \qquad and \qquad \theta = \frac{330(\pi)}{180} = \frac{11}{6}\pi$$

Ex-14-7: (i) Find $\displaystyle\sum_{k=2}^{5} 2^k = 2^2 + 2^3 + 2^4 + 2^5 = 4 + 8 + 16 + 32 = 60$

Ans: $\displaystyle\sum_{k=2}^{5} 2^k = 2^2 + 2^3 + 2^4 + 2^5 = 4 + 8 + 16 + 32 = 60$

(ii) Find the value of n for which for $2^n = \dfrac{1}{64}$

$Ans: \quad 2^n = \dfrac{1}{64} = \dfrac{1}{2^6} = 2^{-6} \qquad \Rightarrow 2^n = 2^{-6} \qquad \Rightarrow n = -6$

Sketch the curve with equation $y = 2^x$

y

1

$y=2^x$

x
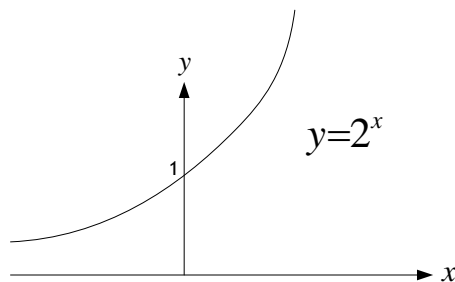
Ex-14-8: The second term of a geometric progression is 18 and the fourth term is 2. The common ratio is positive. Find the sum to infinity of this progression.

Ans:

$$a_n = a_1\, r^{n-1} \qquad \Rightarrow a_2 = a_1 r^{2-1} = a_1 r = 18 \qquad \Rightarrow a_1 r = 18 \quad ...(1)$$

$$and \quad a_4 = a_1 r^{4-1} = a_1 r^3 = 2 \qquad\qquad \Rightarrow a_1 r^3 = 2 \qquad ...(2)$$

Dividing Eq.2 and 1: $\qquad \dfrac{a_1 r^3}{a_1 r} = \dfrac{2}{18} \qquad \Rightarrow r^2 = \dfrac{1}{9} \qquad \Rightarrow r = \dfrac{1}{3}$

$$a_1 r = 18 \qquad \Rightarrow a_1\left(\dfrac{1}{3}\right) = 18 \qquad \Rightarrow a_1 = 54$$

$$S_\infty = \dfrac{a_1}{1-r} = \dfrac{54}{1 - \dfrac{1}{3}} = \dfrac{54}{\dfrac{3-1}{3}} = \dfrac{54}{\dfrac{2}{3}} = \dfrac{3 \times 54}{2} = 81$$

Ex-14-9:   Sketch the graph of $\log_{10} y = 3x + 2$.

$\log_{10} y$

$\log_{10} 10 = 1 = 3x + 2 \Rightarrow x = \dfrac{1}{3}$

$\left(\dfrac{1}{3}, 10\right)$

$\log_{10} y = 3x + 2$

2

0

x
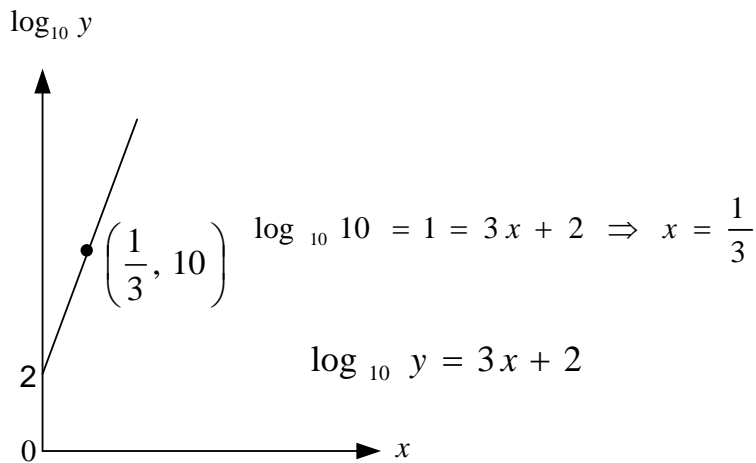
(i)   Find the value of x when y=500, giving your answer correct to 2 decimal places.

Ans: $\log_{10} y = 3x + 2$ $\Rightarrow \log_{10} 500 = 2.699 = 3x + 2$ $\Rightarrow x = 0.233 = 0.23$

(ii)   Find the value of y when x = -1.

Ans:

$$\log_{10} y = 3x + 2 \quad \Rightarrow \log_{10} y = -3 + 2 = -1 \Rightarrow y = 10^{-1} = \dfrac{1}{10}$$

(iii)   Express for $\log_{10}\left(y^4\right)$ in terms of x.

Given: $\log_{10} y = 3x + 2$

$\log_{10} y^4 = 4\log_{10} y = 4(3x + 2) = 12x + 8$

(iv)   Find an expression for y in terms of x.

$\log_{10} y = 3x + 2$ $\Rightarrow y = 10^{3x+2} = 10^2 10^{3x} = 100 \times 10^{3x}$

$\Rightarrow y = 10^{3x+2} = 10^2 10^{3x} = 100 \times 10^{3x}$

$\Rightarrow y = 100 \times 10^{3x}$

Ex-14-10: Fig-14-10 shows a solid cuboid with square base of side x cm and height h cm. Its volume is $120\ cm^3$

h cm   **Fig-14-10**

x cm

x cm
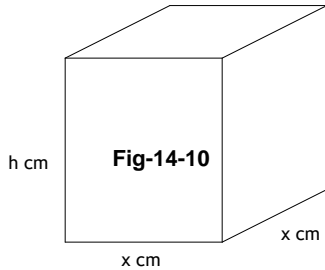
(i) Find h in terms of x. Hence show that the surface area, $A\ cm^2$ .of the cuboid is given by $A = 2x^2 + \dfrac{480}{x}$

Ans: The volume: $V = x^2 h = 120 \qquad \Rightarrow h = \dfrac{120}{x^2}$

Ans: The surface area:

$$x^2 + x^2 + hx + hx + hx + hx = 2x^2 + 4hx = 2x^2 + 4x\left(\dfrac{120}{x^2}\right) = 2x^2 + \dfrac{480}{x}$$

(ii) Find $\dfrac{dA}{dx}$ and $\dfrac{d^2A}{dx^2}$

$$A = 2x^2 + \dfrac{480}{x} = 2x^2 + 480\,x^{-1} \qquad \Rightarrow \dfrac{dA}{dx} = 4x - 480\,x^{-2} \qquad and \qquad \dfrac{d^2A}{dx^2} = 4 + 960\,x^{-3}$$

$$\Rightarrow \dfrac{dA}{dx} = 4x - \dfrac{480}{x^2}$$

$$and \qquad \dfrac{d^2A}{dx^2} = 4 + \dfrac{960}{x^3}$$

(iii) Hence find the value of x which gives the minimum surface area. Find also the value of the surface area in this case.

For maximum area,

$$\Rightarrow \dfrac{dA}{dx} = 4x - \dfrac{480}{x^2} = \dfrac{4x^3 - 480}{x^2} = 0 \quad \Rightarrow 4x^3 = 480 \qquad \Rightarrow x = \sqrt[3]{120} = 4.932$$

$$and \qquad A = 2x^2 + \dfrac{480}{x}\Bigg|_{x=4.932} = 2(4.932)^2 + \dfrac{480}{4.932} = 48.658 + 97.323 = 145.9$$

Ex-14-11:   (i)   The course for a yacht race is a triangle (see Fig-14-11-1).

The yachts start at A, then travel to B, then to C and finally back to A.

Fig-14-11-1
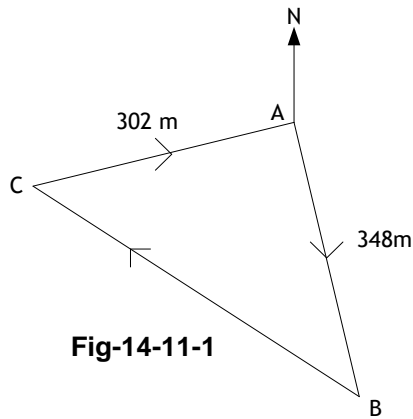
(A)   Calculate the total length of the course for this race.

(B)   Give that the bearing of the first stage, AB, is 175^0, calculate the bearing of the second stage, BC.

Ans:

Find $BC^2 = (302)^2 + (348)^2 - 2(302)(348)\cos 72^0 = 147356.18 \implies BC = 383.87\, m$

The total length of the course=302+348+383.87=1033.87m

and $\dfrac{302}{\sin B} = \dfrac{383.87}{\sin 72^0} \implies \sin B = \dfrac{302\sin 72^0}{383.87} = 0.74826 \implies B = 48.44^0$

Bearing of BC=360^0-5^0-48.44^0=306.56^0
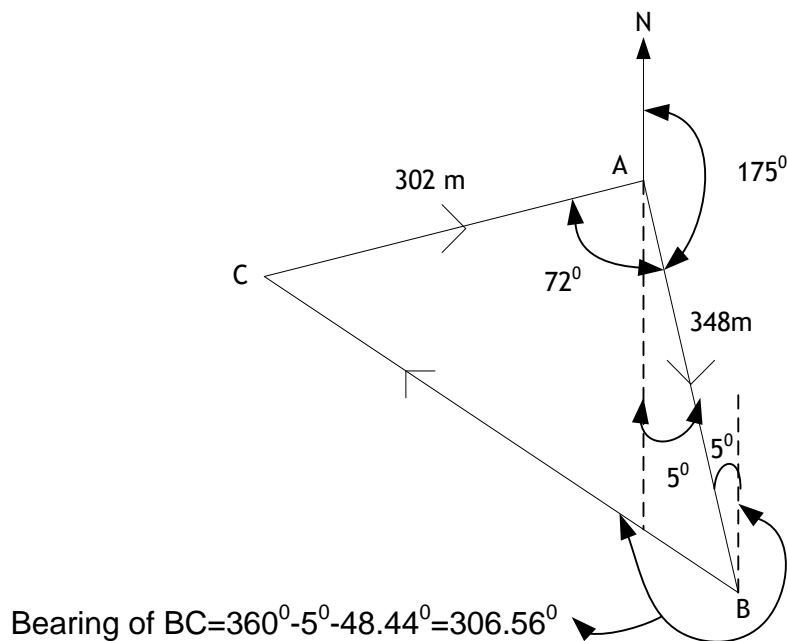
(ii) Fig-14-11-2 shows the course of another yacht race. The course follows the arc of a circle from P to Q, then a straight line back to P. The circle has radius 120 m and centre O; angle POQ $=136^0$

$360^0 - 136^0 = 224^0$
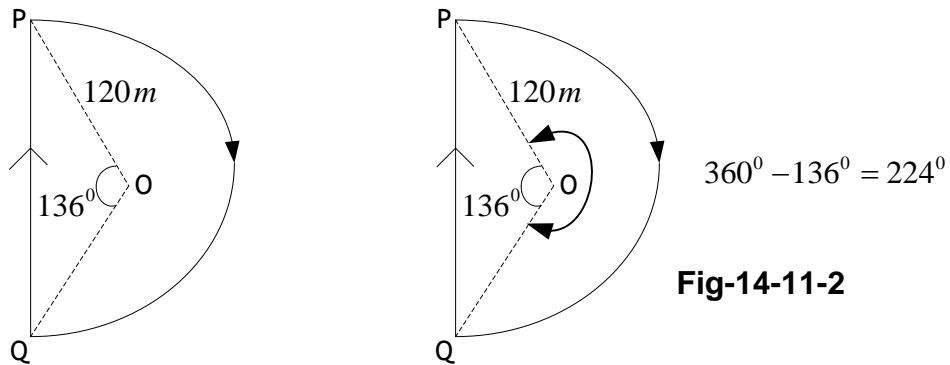
Fig-14-11-2

Calculate the total length of the course for this race.

The length of the arc $S = r\theta = 120\left(\dfrac{224\pi}{180}\right) = 120(3.91) = 469.14m$

Find $PQ^2 = (120)^2 + (120)^2 - 2(120)(120)\cos 136^0 = 49284 \Rightarrow PQ = 222\ m$

The total length of the course=PQ+S=222+469.14=691.14

Ex-14-12: (i) Fig-14-12 shows part of the curve $y = x^4$ and the line $y = 8x$, which intersect at the origin and the point P.

Fig-14-12
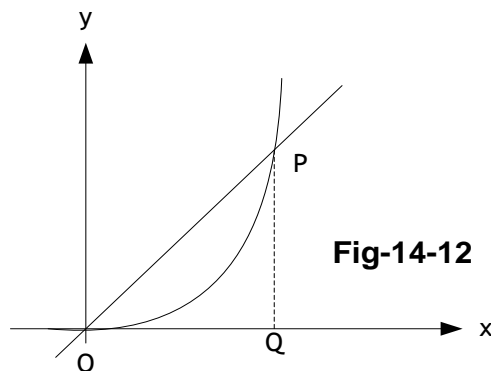
(A) Find the coordinates of P, and show that the area of triangle OPQ is 16 square units.

Ans: At point P, the values of y are the same:
$x^4 = 8x \Rightarrow x(x^3 - 8) = 0 \Rightarrow x = 0,\ and\ x^3 = 8 \Rightarrow x = 2 \Rightarrow P = (2,16)$

Area of the triangle OPQ=: $\dfrac{1}{2}(OQ \times PQ) = \dfrac{1}{2}(2 \times 16) = 16\ square\ unit$

(B)   Find the area of the region bounded by the line and the curve.
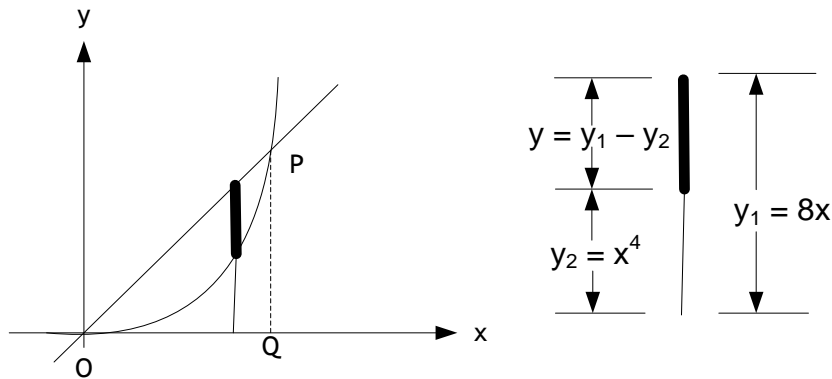
$$\int_0^2 (y_1 - y_2)\,dx = \int_0^2 \left(8x - x^4\right)dx = \frac{8x^2}{2} - \frac{x^5}{5}\Big|_0^2 = 4x^2 - \frac{x^5}{5}\Big|_0^2 = 4(2)^2 - \frac{2^5}{5} - 0 - 0 = 16 - \frac{32}{5} = \frac{48}{5} = 9.6$$

(ii)   You are given that $f(x) = x^4$

(A)   Complete this identity for $f(x+h)$

$$f(x+h) = (x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$$

(B)   Simplify

$$\frac{f(x+h) - f(x)}{h} = \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} = \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} = 4x^3 + 6x^2h + 4xh^2 + h^3$$

(C)   Find $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\left(4x^3 + 6x^2h + 4xh^2 + h^3\right) = 4x^3$

(D)   State what this limit represents.

   Ans: This limit represents $\dfrac{df}{dx}$

PAPER-15

Ex-15-1: Find $\int \left(20x^4 + 6x^{-\frac{3}{2}} \right) dx = \frac{20x^5}{5} + 6(-2)x^{-\frac{1}{2}} + C = 4x^5 - 12x^{-\frac{1}{2}} + C$

Ex-15-2: Fig-15-2 shows the coordinates at certain points on a curve.

Fig-15-2
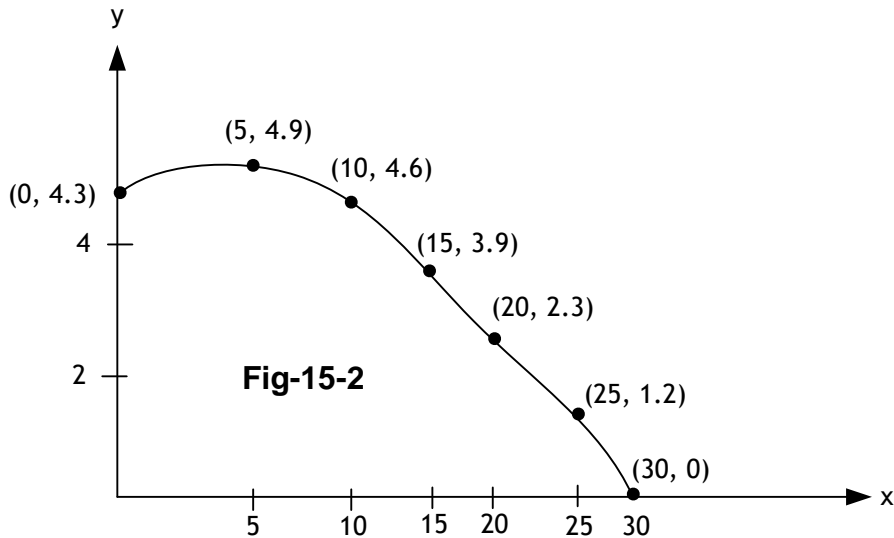
Use the trapezium rule with 6 strips to calculate an estimate of the area of the region bounded by this curve and axes.

Ans: $Area \approx \frac{h}{2} [y_0 + y_6 + 2(y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5)]$ *from Formula sheet*

$$h = \frac{b-a}{n} = \frac{30-0}{6} = 5$$

$$Area \approx \frac{5}{2}[4.3 + 0 + 2(4.9 + 4.6 + 3.9 + 2.3 + 1.2)] = 2.5(38.1) = 95.25$$

Ex-15-3: Find $\sum_{k=1}^{5} \frac{1}{1+k}$

Ans: $\sum_{k=1}^{5} \frac{1}{1+k} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{30 + 20 + 15 + 12 + 10}{60} = \frac{87}{60} = \frac{29}{20} = 1.45$

Ex-15-4: Solve the equation $\sin 2x = -0.5$ for $0^0 \leq x \leq 180^0$

Ans: $\sin 2x = -0.5 \quad \Rightarrow 2x = \sin^{-1}(-0.5) = -30^0 (i.e.\ 210^0, 330^0) \quad \Rightarrow x = 105^0, 165^0$

Ex-15-5:   Fig.5 shows the graph of $y = f(x)$
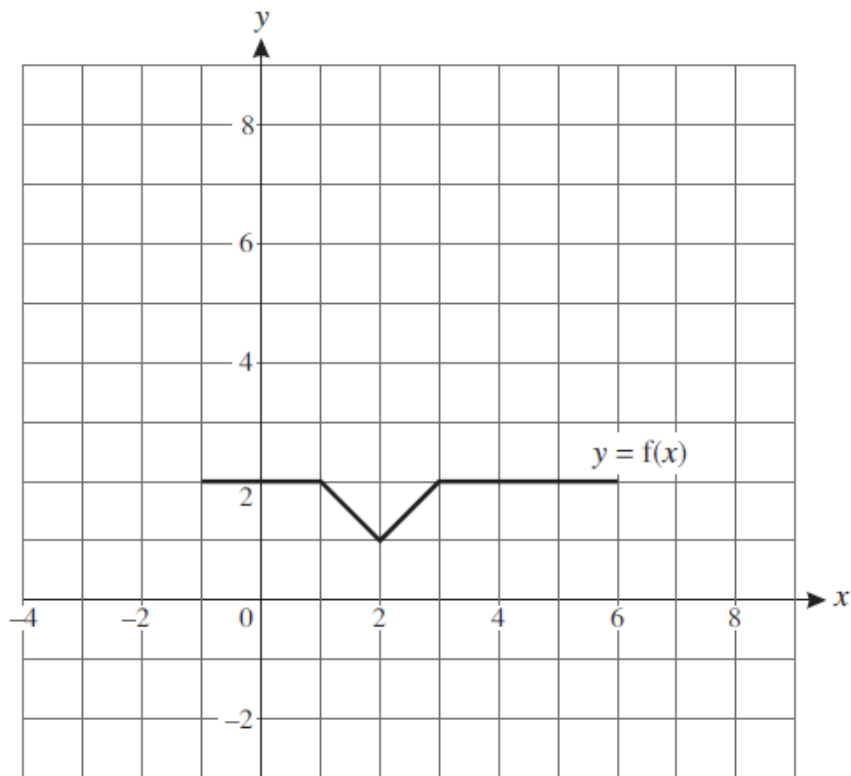
Fig. 5

On the insert, draw the graph of

(i) $y = f(x - 2)$

(ii) $y = 3f(x)$

(i) $y = f(x - 2)$

(4,1)

(ii) $y = 3f(x)$

(2,3)
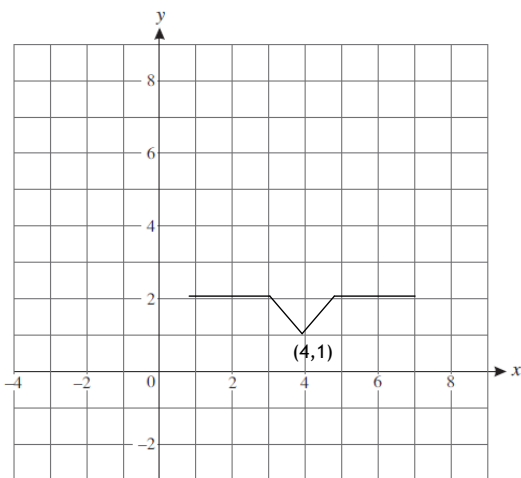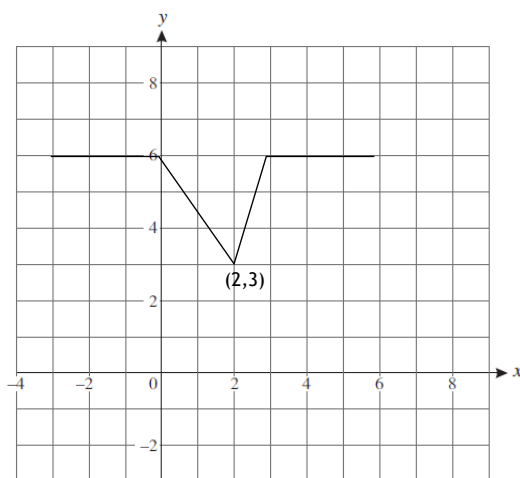
Ex-15-6: An arithmetic progression has first term 7 and third term 12

    (i) Find the 20th term of this progression.

    Ans: $a_n = a_1 + (n-1)d = 7 + (n-1)d$ $\Rightarrow a_n = 7 + (n-1)d$

$$\Rightarrow a_3 = 7 + (3-1)d = 12 \quad \Rightarrow 2d = 12 - 7 = 5 \quad \Rightarrow d = 2.5$$

$$\Rightarrow a_{20} = 7 + (20-1)(2.5) = 7 + 47.5 = 54.5$$

    (ii) Find the sum of the 21st to the 50th terms inclusive of this progression.

    Ans:

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{30}{2}(a_{21} + a_{50})$$

$$\Rightarrow a_{21} = 7 + (21-1)(2.5) = 7 + 50 = 57$$

$$\Rightarrow a_{50} = 7 + (50-1)(2.5) = 7 + 122.5 = 129.5$$

$$S_{30} = \frac{30}{2}(a_{21} + a_{50}) = 15(57 + 129.5) = 2797.5$$

Ex-15-7: Differentiate $4x^2 + \dfrac{1}{x}$ and hence find the x-coordinate of the stationary point of the curve $y = 4x^2 + \dfrac{1}{x}$

    Ans: $y = 4x^2 + \dfrac{1}{x} = 4x^2 + x^{-1}$ $\Rightarrow \dfrac{dy}{dx} = 8x - \dfrac{1}{x^2}$

For stationary point: $\dfrac{dy}{dx} = 8x - \dfrac{1}{x^2} = 0$ $\Rightarrow \dfrac{8x^3 - 1}{x^2} = 0$ $\Rightarrow x^3 = \dfrac{1}{8}$ $\Rightarrow x = \dfrac{1}{2}$

Ex-15-8: The terms of a sequence are given by:

$$u_1 = 192$$

$$u_{n+1} = -\frac{1}{2}u_n$$

    (i) Find the third term of this sequence and state what type of sequence it is.

Ans:

$$u_1 = 192$$

$$u_2 = -\frac{1}{2}u_1 = -\frac{192}{2} = -96$$

$$u_3 = -\frac{1}{2}u_2 = \frac{96}{2} = 48$$

It is a geometric series with the ratio:

$$r = -\frac{1}{2}$$

(ii) Show that the series $u_1 + u_2 + u_3 + \dots$ converges and find its sum to infinity.

Because : $|r| < 1$

The series will converge

$$S_\infty = \frac{a_1}{1-r} = \frac{192}{1+\dfrac{1}{2}} = 128$$

Ex-15-9:   (i)   State the value of $\log_a a$

Ans: $\log_a a = 1$

(ii) Express each of the following in terms of $\log_a x$

(A) $\log_a x^3 + \log_a \sqrt{x}$

Ans: $\log_a x^3 + \log_a \sqrt{x} = 3\log_a x + \frac{1}{2}\log_a x = \frac{7}{2}\log_a x$

(B) $\log_a \dfrac{1}{x}$

Ans: $\log_a \dfrac{1}{x} = \log_a x^{-1} = -\log_a x$

Ex-15-10: Fig-15-10 shows a sketch of the graph of $y = 7x - x^2 - 6$

Fig-15-10
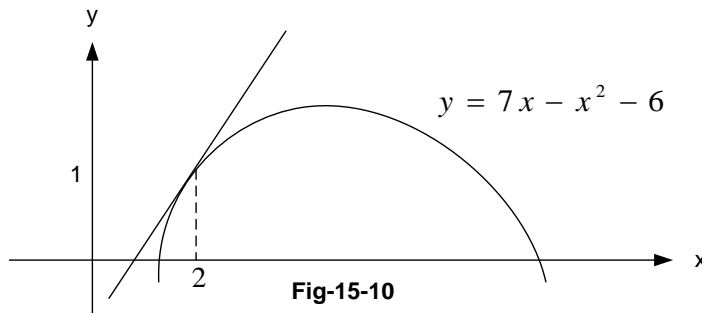
(i) Find $\dfrac{dy}{dx}$ and hence find the equation of the tangent to the curve at the point on the curve where $x = 2$

Ans: $\dfrac{dy}{dx} = 7 - 2x \Big|_{x=2} = 7 - 4 = 3 = m = gradient\, of\, the \tan gent\, at\, (2,4)$

$y = mx + c = 3x + c \qquad \Rightarrow 4 = 3(2) + c \qquad \Rightarrow c = 4 - 6 = -2 :$

$y = 3x - 2$

(ii) Show the curve crosses the x-axis where x = 1 and find the x-coordinate of the other point of intersection of the curve with the x-axis.

Ans: When the curve crosses the x-axis, y=0

$0 = 7x - x^2 - 6 \quad \Rightarrow x^2 - 7x + 6 = (x - 6)(x - 1) = 0 \quad \Rightarrow x = 1 \quad and \quad x = 6$

Hence, points of intersections are: P1(1,0) and P2(6,0)

(iii) Find $\displaystyle\int_{1}^{2}(7x - x^2 - 6)dx$

$$\int_{1}^{2}(7x - x^2 - 6)dx = \frac{7x^2}{2} - \frac{x^3}{3} - 6x \Big|_{1}^{2} = \left(\frac{7 \times 2^2}{2} - \frac{2^3}{3} - 12\right) - \left(\frac{7 \times 1^2}{2} - \frac{1^3}{3} - 6\right) = 2 - \frac{8}{3} - \frac{7}{2} + \frac{1}{3} + 6 = \frac{13}{6}$$

Hence find the area of the region bounded by the curve, the tangent and the x-axis, shown shaded on Fig.10.
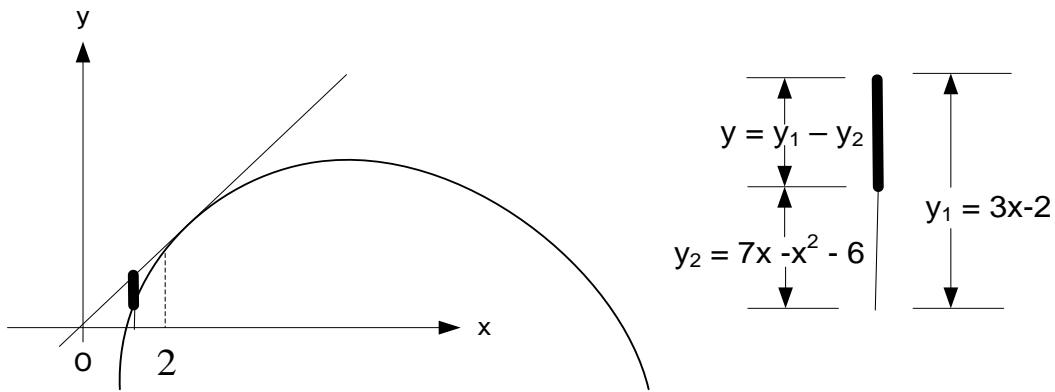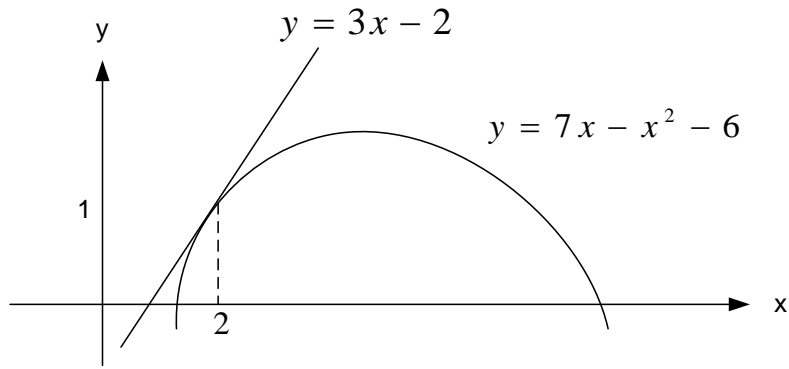
$$Area = \int_0^2 (y_1 - y_2)dx = \int_0^2 \left(3x - 2 - 7x + x^2 + 6\right)dx = \int_0^2 \left(4 - 4x + x^2\right)dx = 4x - \frac{4x^2}{2} + \frac{x^3}{5}\Big|_0^2$$

$$Area = 4x - \frac{4x^2}{2} + \frac{x^3}{5}\Big|_0^2 = 4(2) - \frac{4 \times 2^2}{2} + \frac{2^3}{5} - 0 - 0 - 0 = 8 - 8 + \frac{8}{3} = \frac{8}{3}$$

Ex-15-11:   (i)   Fig-15-11 shows the surface ABCD of a TV presenter's desk. AB and CD are arcs of circles with centre O and sector angle 2.5 radians. OC = 60 cm and OB = 140 cm.
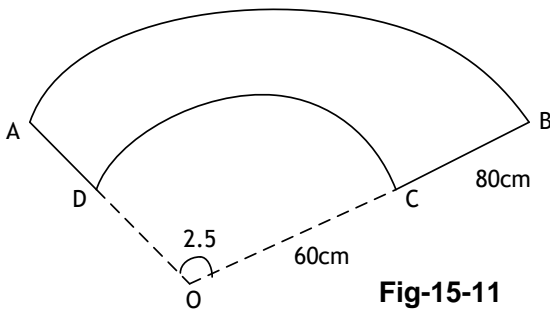
Fig-15-11

Calculate the length of arc CD
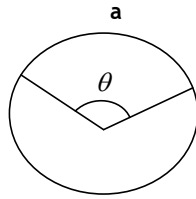
And the length of the arc is **a = r θ**

$$CD = r\theta = 60(2.5) = 150 cm$$

(A)   Calculate the area of the surface ABCD of the desk

$$\text{Area} = = \frac{1}{2}r_1^2\theta\,(radian) - \frac{1}{2}r_2^2\theta\,(radian) = \frac{1}{2}\left(r_1^2 - r_2^2\right)\theta = \frac{1}{2}\left(80^2 - 60^2\right)(2.5) = 4500\ cm^2$$

(ii)   The TV presenter is at point P, shown in Fig-15-11-1. A TV camera can move along the track EF, which is of length 3.5 m.
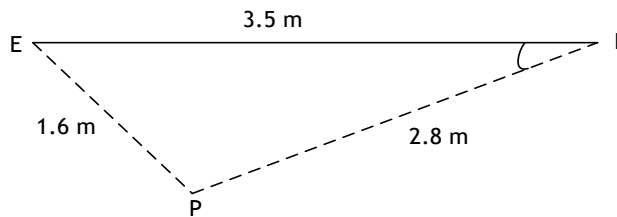
Fig-15-11-1

When the camera is at E, the TV presenter is 1.6 m away. When the camera is at F, the TV presenter is 2.8m away.

(A)   Calculate, in degrees, the size of angle EFP.

Ans:

$$EP^2 = EF^2 + FP^2 - 2(EF)(FP)\cos(F) \Rightarrow \cos F = \frac{FF^2 + FP^2 - EP^2}{2(EF)(FP)} = \frac{(3.5)^2 + (2.8)^2 - (1.6)^2}{2(3.5)(2.8)}$$

$$\cos F = 0.8944 \quad \Rightarrow F = \cos^{-1}(0.8944) = 26.57^0$$

(B)   Calculate the shortest possible distance between the camera and the TV presenter.
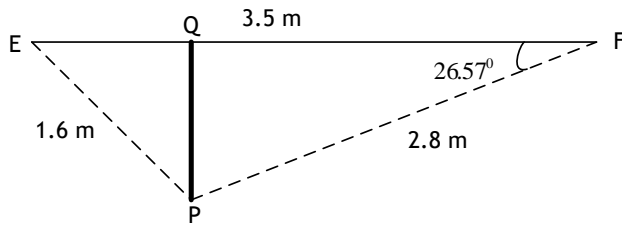
The shortest distance from the camera to P is PQ as:

$$\sin 26.57^0 = \frac{PQ}{2.8} \Rightarrow PQ = 2.8 \sin 26.57^0 = 1.2524 \ m$$

Ex-15-12: Answer part (ii) of this question on the insert provided.

The proposal for a major building project was accepted, but actual construction was delayed. Each year a new estimate of the cost was made. The table shows the estimated cost, £y million, of the project t years after the project was first accepted.

Years after proposal accepted(t)	1	2	3	4	5
Cost (£y million)	250	300	360	440	530

The relationship between y and t is modeled by $y = ab^t$, where a and b are constants.

(i) Show that $y = ab^t$ may be written as $\log_{10} y = \log_{10} a + t \log_{10} b$

Ans: $y = ab^t \Rightarrow \log_{10} y = \log_{10} ab^t = \log_{10} a + \log_{10} b^t = \log_{10} a + t \log_{10} b$

$\Rightarrow \log_{10} y = \log_{10} a + t \log_{10} b$

(ii) On the insert, complete the table and $\log_{10} y$ plot against t,

drawing by eye a line of best fit.

Years after proposal accepted (t)	1	2	3	4	5
Cost (£y million)	250	300	360	440	530
$\log_{10} y$	2.398	2.477	2.556	2.643	2.724
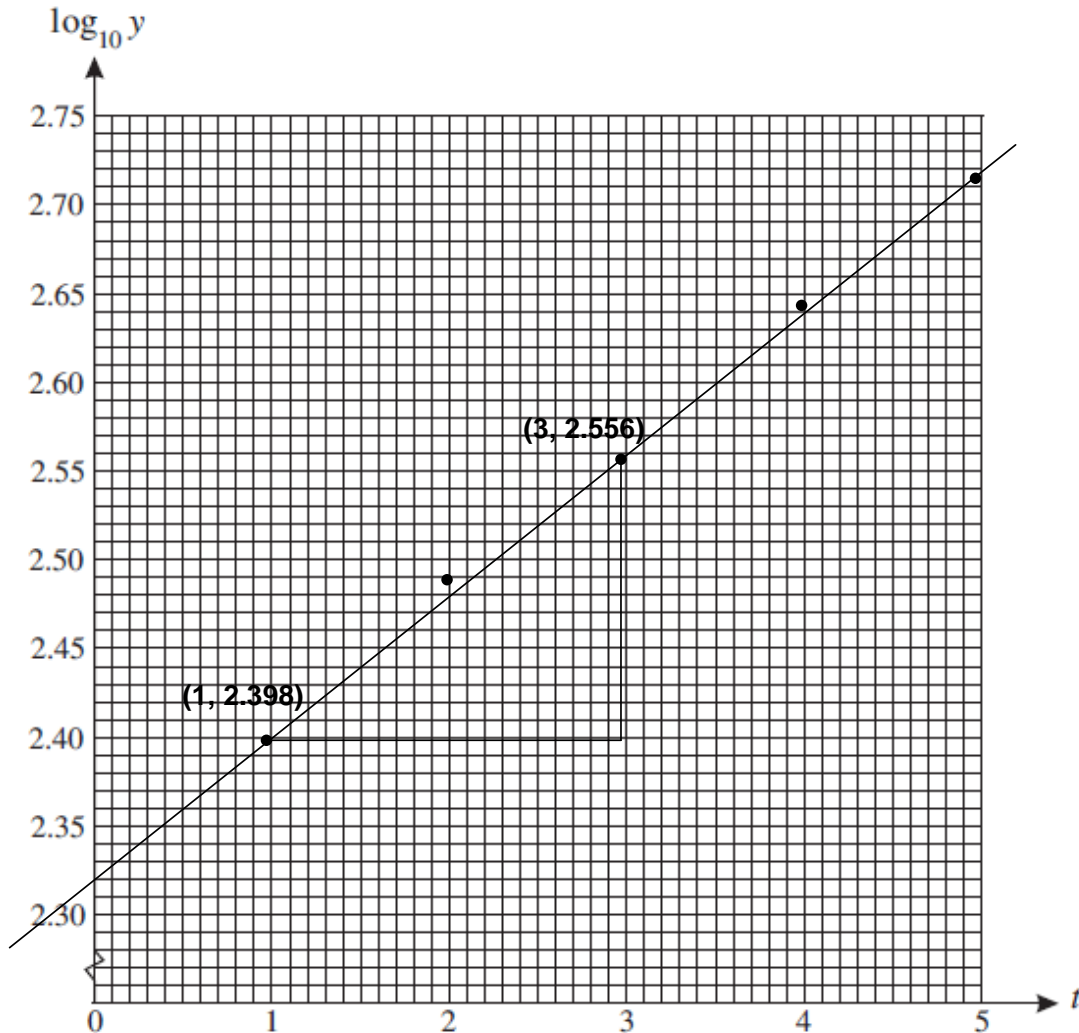
(iii)   Use your graph and the result of part (i) to find the values
of $\log_{10} a$ and $\log_{10} b$, and hence **a** and **b**.

$$\log_{10} b = gradient = \frac{2.556 - 2.398}{3 - 1} = 0.079 \Rightarrow b = 10^{0.079} = 1.199 \cong 1.2$$

$$\log_{10} a = y - \text{int } ercept = 2.32 \Rightarrow a = 208.93$$

(i)   According to model, what was the estimated cost of the project when
it was first accepted?

$$\log_{10} y = \log_{10} a + t \log_{10} b$$

$$\log_{10} y = 2.32 + 0.079t \Rightarrow \log_{10} y \Big|_{t=0} = 2.32 + 1.2t = 2.32 \Rightarrow y = 208.92$$

(ii)   Find the value of t given by this model when the estimated cost is £1000 million. Give your answer rounded to 1 decimal place.

$$\log_{10} y = \log_{10} a + t \log_{10} b$$

$$\log_{10} y = 2.32 + 0.079t \Rightarrow \log_{10}(1000) = 3 = 2.32 + 0.079t \Rightarrow t = \frac{3 - 2.32}{0.079} = 8.61$$

PAPER-16

SECTION A (C2-12-1-05)

Ex-16-1:   Find $\dfrac{dy}{dx}$ when $y = x^6 + \sqrt{x}$

Ans: $y = x^6 + \sqrt{x} = x^6 + x^{\frac{1}{2}} \quad \Rightarrow \dfrac{dy}{dx} = 6x^5 + \dfrac{1}{2}x^{-\frac{1}{2}} = 6x^5 + \dfrac{1}{2\sqrt{x}}$

Ex-16-2:   Find $\displaystyle\int\left(x^3 + \dfrac{1}{x^3}\right)dx$

Ans: $\displaystyle\int\left(x^3 + \dfrac{1}{x^3}\right)dx = \int\left(x^3 + x^{-3}\right)dx = \dfrac{x^{3+1}}{3+1} + \dfrac{x^{-3+1}}{-3+1} = \dfrac{x^4}{4} - \dfrac{x^{-2}}{2} = \dfrac{x^4}{4} - \dfrac{1}{2x^2} + C$

Ex-16-3:   Sketch the graph of $y = \sin x \ \ for \ \ 0^0 \le x \le 360^0$

Ans:
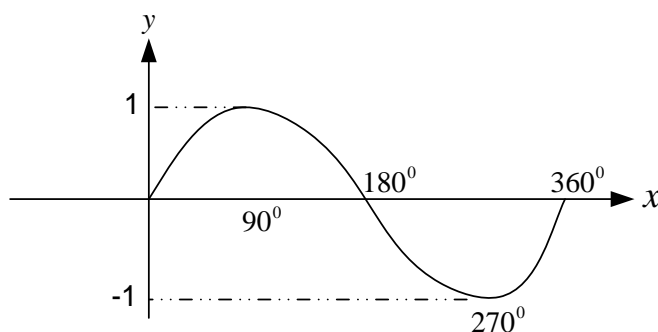
Solve the equation $\sin x = -0.2 \ \ for \ \ 0^0 \le x \le 360^0$

Ans: $\sin x = -0.2 \quad \Rightarrow x = \sin^{-1}(-0.2) = 191.54^0, \ 348.46^0$

Ex-16-4:   For triangle ABC shown in Fig-16-4, calculate

B

6.8cm

A 108^0

4.1cm   Fig-16-4

C
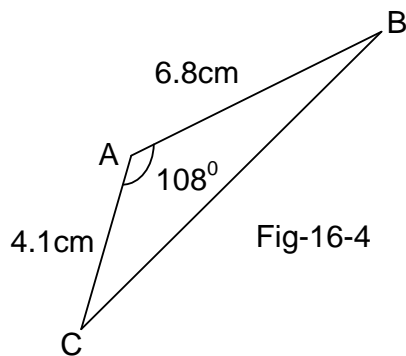
(i)   The length of BC,
(ii)   The area of triangle ABC.

Ans: Because 2 sides and the angle between them is given, then cosine rule is used:

B

c=6.8cm

A 108^0
a

b=4.1cm

C
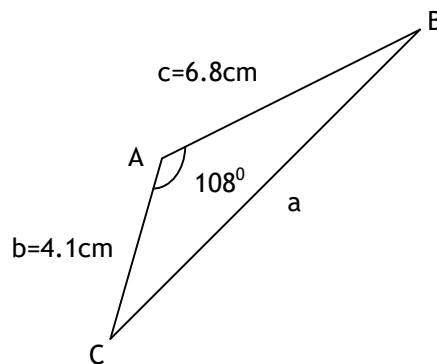
$a^2 = b^2 + c^2 - 2bc\cos A = (4.1)^2 + (6.8)^2 - 2 \times 4.1 \times 6.8\cos(108^0) = 16.81 + 46.24 - 55.76(-0.309)$
$\Rightarrow a^2 = 80.28 \quad \Rightarrow a = 8.96\, cm$

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} \quad \Rightarrow \dfrac{8.96}{\sin(108^0)} = \dfrac{4.1}{\sin B} = \dfrac{6.8}{\sin C} \quad \Rightarrow \sin B = \dfrac{4.1 \times \sin(108^0)}{8.96} = 0.4576$
$\Rightarrow B = 27.232^0$

$C = 180 - (108^0 + 27.232) = 44.768^0$

$Area = \dfrac{1}{2}b \times c \times \sin A = \dfrac{1}{2}\left(4.1 \times 6.8 \times \sin(108^0)\right) = 13.258\, cm^2$

Note: Area of a triangle

The formula always uses 2 sides and the angle formed by those sides:
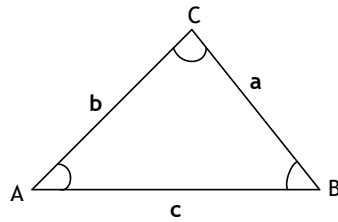
$$Area = \frac{1}{2}bc\sin A = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B$$

Ex-16-5:   The first three terms of a geometric progression are 4, 2, 1.

Find the 20th term, expressing your answer as a power of 2.

Ans:

$$a_n = a_1 r^{n-1} \Rightarrow a_{20} = a_1 r^{20-1} = 4r^{19} \quad and \quad r = \frac{2}{4} = \frac{1}{2} \Rightarrow a_{20} = 4\left(\frac{1}{2}\right)^{19} = (2)^2\left(2^{-19}\right) = 2^{-17}$$

Find also the sum to infinity of this progression.

Ans: $S = \dfrac{a_1}{1-r} \quad for \ \inf inite \ GP \quad \Rightarrow S = \dfrac{a_1}{1-r} = \dfrac{4}{1-0.5} = \dfrac{4}{0.5} = 8$

Ex-16-6:   A sequence is given by:

$$a_1 = 4$$

$$a_{r+1} = a_r + 3$$

Write down the first 4 terms of this sequence.

Ans: $a_1 = 4$

$$a_{1+1} = a_2 = a_1 + 3 = 4 + 3 = 7$$

$$a_{2+1} = a_3 = a_2 + 3 = 7 + 3 = 10$$

$$a_{3+1} = a_4 = a_3 + 3 = 10 + 3 = 13$$

$$a_{4+1} = a_5 = a_4 + 3 = 13 + 3 = 16$$

Find the sum of the first 100 terms of the sequence.

Ans: It is an arithmetic series as the difference, d=13-10=3

$$S_n = \frac{n}{2}(a+l) \quad S_{100} = \frac{100}{2}(4+l) \quad and \quad l = a + (n-1)d = 4 + (100-1)(3) = 4 + 297 = 301$$

$$\Rightarrow S_{100} = \frac{100}{2}(4+301) = \frac{30500}{2} = 15250$$

Ex-16-7: Fig-16-7 shows a sector of a circle of radius 5 cm which has angle θ radians. The sector has area of $30\,cm^2$

5cm θ
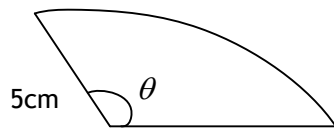
Fig-16-7

(i)   Find θ.

Ans: A=Area

$$\frac{1}{2}r^2\theta\,(radian) \Rightarrow \theta = \frac{2A}{r^2} = \frac{2(30)cm^2}{25cm^2} = 2.4 \ radians = \frac{2.4 \times 180}{\pi} = 137.51^0$$

(ii)   Hence find the perimeter of the sector.

a

a

θ
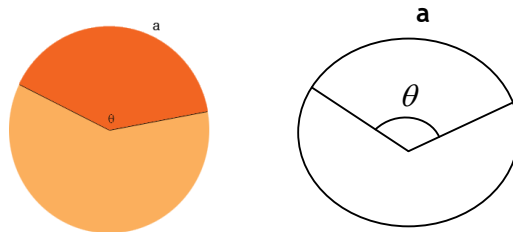
And the length of the arc is **a = r θ**

$$a = r\theta = 5(2.4) = 12\,cm \qquad P = perimeter = r + r + a = 5 + 5 + 12 = 22\,cm$$

Ex-16-8:   (i)   Solve the equation $10^x = 316$

Ans: $\log(10^x) = \log(316) \qquad \Rightarrow x\log 10 = x = \log(316) = 2.4997$

(II) Simplify $\log_a(a^2) - 4\log_a\left(\frac{1}{a}\right)$

Ans

$$\log_a(a^2) - 4\log_a\left(\frac{1}{a}\right) = \log_a(a^2) - 4\log_a(a)^{-1} = 2\log_a a + 4\log_a a = 6\log_a a = 6$$

Ex-16-9: (i) A tunnel is 100m long. Its cross-section, shown in Fig-16-9-1, is modelled by the curve $y = \frac{1}{4}\left(10x - x^2\right)$,

Where x and y are horizontal and vertical distances in meters.

Fig-16-9-1
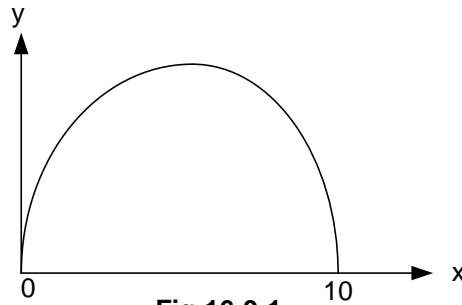
Using this model,

(A) Find the greatest height of the tunnel,

Ans: For maximum value: $\frac{dy}{dx} = 0$ *and* $\frac{d^2y}{dx^2} < 0$

$y = \frac{1}{4}\left(10x - x^2\right)$ $\Rightarrow \frac{dy}{dx} = \frac{1}{4}\left(10 - 2x\right) = 0$ $\Rightarrow 10 - 2x = 0$ $\Rightarrow x = 5$,

and $\frac{d^2y}{dx^2} = -\frac{1}{2} < 0$ $\Rightarrow y_{max} = \frac{1}{4}\left(10 \times 5 - 5^2\right) = \frac{1}{4}\left(50 - 25\right) = \frac{25}{4} = 6.25$

(B) Explain why $100\int_{0}^{10} ydx$ gives the volume, cubic meters, of earth removed to make the tunnel. Calculate this volume.

Ans: $\int_{0}^{10} ydx = Area\ under\ the\ curve$

And if area is multiplied by the height, then it will give volume.

$100\int_{0}^{10} ydx = 100\int_{0}^{10}\left(\frac{1}{4}\left(10x - x^2\right)\right)dx = \frac{100}{4}\int_{0}^{10}\left(10x - x^2\right)dx = 25\int_{0}^{10}\left(10x - x^2\right)dx$

$\Rightarrow Volume = 25\left(\frac{10x^2}{2} - \frac{x^3}{3}\right)\Big|_{0}^{10} = 25\left(\frac{10 \times 10^2}{2} - \frac{10^3}{3}\right) = 25\left(\frac{3000 - 2000}{6}\right) = 4166.67\ cm^3$

(II) The roof the tunnel is re-shaped to allow for larger vehicles. Fig-16-9-2 shows the new cross-section
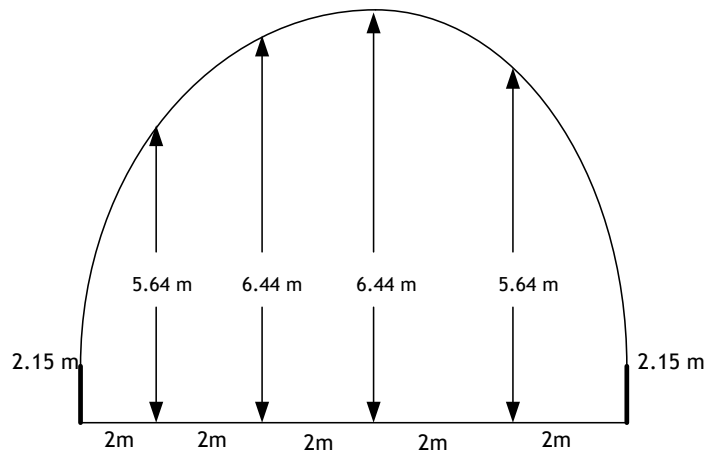
Use the trapezium rule with 5 strips to estimate the new cross sectional area.

Hence, estimate the volume of earth removed when the tunnel is reshaped.

Ans:

$$y = \frac{1}{4}\left(10x - x^2\right) \qquad h = \frac{b-a}{n} = \frac{10-0}{5} = 2m$$
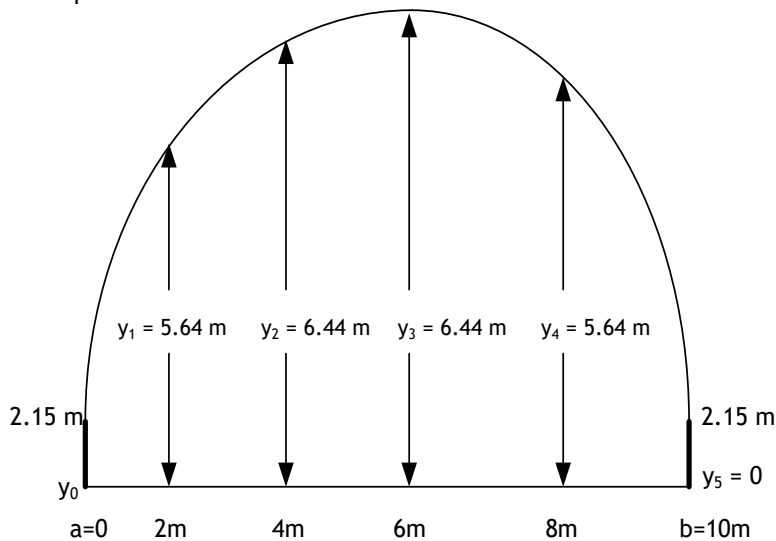
$$\Rightarrow \int_0^{10} \frac{1}{4}\left(10x - x^2\right)dx \approx \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + y_5\right)$$

$$\Rightarrow \int_0^{10} \frac{1}{4}\left(10x - x^2\right)dx \approx \frac{2}{2}\left[2.15 + 2(5.64 + 6.44 + 6.44 + 5.64) + 2.15\right] = 52.62\,m^2$$

Volume $= 100(52.62) = 5262$ $\qquad \Rightarrow$ Volume Re$moved = 5262 - 4167 = 1095\,m^3$.

Ex-16-10:   A curve has equation $y = x^3 - 6x^2 + 12$.

(i)   Use calculus to find the coordinates of the turning points of this curve. Determine also the nature of these turning points.

Ans: For turning point or (maximum/minimum), the following should apply.

$$\frac{dy}{dx} = gradient = 0 \quad and \quad \frac{d^2y}{dx^2} < 0 \quad for \; Maximum$$

$$\frac{dy}{dx} = gradient = 0 \quad and \quad \frac{d^2y}{dx^2} > 0 \quad for \; Minimum$$

$$y = x^3 - 6x^2 + 12 \quad \Rightarrow \frac{dy}{dx} = 3x^2 - 12x = 0 \quad \Rightarrow 3x(x-4) = 0 \quad \Rightarrow x = 0 \quad and \; x = 4$$

$$and \quad \frac{d^2y}{dx^2} = 6x - 12 \bigg|_{x=0} = -12 \quad and \quad \frac{d^2y}{dx^2}\bigg|_{x=4} = 24 - 12 = 12$$

Hence, there are 2 turning points: Max at (0,12) and Min at (4,-20)

(ii)   Find in the form $y = mx + c$, the equation of the normal to the curve at the point (2, -4).

Ans:

$$\frac{dy}{dx} = 3x^2 - 12x \bigg|_{x=2} = 3(2)^2 - 12(2) = 12 - 24 = -12 = gradient \, of \, the \tan gent \, at \, (2,-4)$$

Hence, the gradient of the normal is $m_2 = -\dfrac{1}{m_1} = \dfrac{1}{12}$

$$y = mx + c = \frac{1}{12}x + c \quad \Rightarrow -4 = \frac{1}{12}(2) + c \quad \Rightarrow c = -4 - \frac{1}{6} = -\frac{25}{6} :$$

$$y = \frac{1}{12}x - \frac{25}{6}$$

Ex-16-11:   Answer part (iii) of this question on the insert provided.

A hot drink is made and left to cool. The table shows its temperature at ten-minute intervals after it is made.

Time (minutes)	10	20	30	40	50
Temprature (C^0)	68	53	42	36	31

The room temperature is $22C^0$. The difference between the temperature of the drink and room temperature at time t minutes is zC^0. The relationship between z and t is modeled by $z = z_0 10^{-kt}$ where z_0 and k are positive constants.

(i) Give a physical interpretation for constant z_0

(ii) Show that $\log_{10} z = -kt + \log_{10} z_0$

Ans: $z = z_0 10^{-kt}$ $\Rightarrow \log_{10} z = \log_{10}\left(z_0 10^{-kt}\right) = \log_{10} z_0 + \log_{10}(10)^{-kt}$

$\log_{10} z_0 - kt\log_{10} 10 = \log_{10} z_0 - kt$

On the insert, complete the table and draw the graph of $\log_{10} z$ against t.

Use your graph to estimate the values of k and z_0

Hence estimate the temperature of the drink 70 minutes after it is made.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1.55 - 1.0}{50 - 20} = \frac{0.55}{30} = 0.018$$

t	10	20	30	40	50
z	46	33	22	16	11
logz	1.662	1.518	1.342	1.204	1.041

$\log_{10} z = -kt + \log_{10} z_0$

$\log_{10} z_0 =$ y-intercept=1.9

$Z_0 = \log^{-1}(1.9) = 79.43$
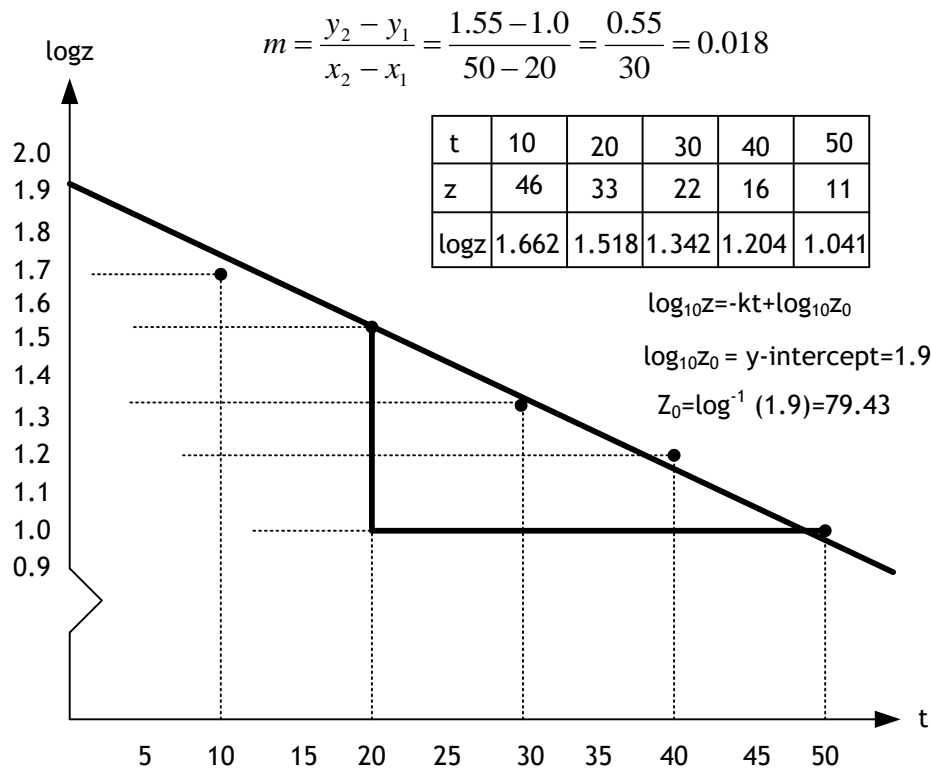
$z = z_0 10^{-kt} = 79.43e^{-0.018t}$

$z\big|_{t=70} = 79.43e^{-0.018(70)} = 79.43e^{-1.26} = 79.43(0.283) = 22.53$

t	10	20	30	40	50
z	46	33	22	16	11
$\log_{10} z$	1.662	1.518	1.342	1.204	1.041
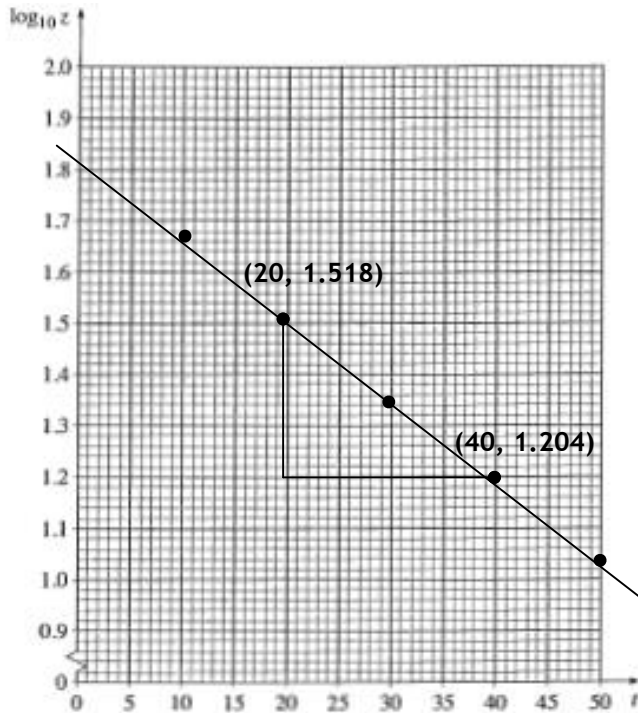
$\log_{10}z=-kt+\log_{10}z_0$

$\log_{10}z_0$ = y-intercept=1.8

$Z_0=\log^{-1}(1.8)=63.1$

$$k = grasdient = \frac{1.204 - 1.518}{40 - 20} = -0.0157 \Rightarrow k = 0.0157$$

PAPER-17

SECTION A (C2-16-06)

Ex-17-1: Given that $140^0 = k\pi$ radians, find the exact value of k.

Ans: $140^0 = \dfrac{140^0 \times \pi}{180} = \dfrac{7\pi}{9} = k\pi \qquad \Rightarrow k = \dfrac{7}{9}$

Ex-17-2: Find the numerical value of $\displaystyle\sum_{k=2}^{5} k^3$

Ans: $\displaystyle\sum_{k=2}^{5} k^3 = 2^3 + 3^3 + 4^3 + 5^3 = 8 + 27 + 64 + 125 = 224$

Ex-17-3: Beginning with the triangle shown in Fig-17-3, prove that that

$$\sin 60^0 = \frac{\sqrt{3}}{2}.$$

Fig-17-3

Ans:
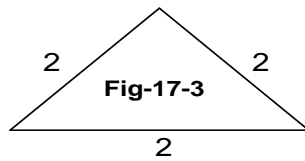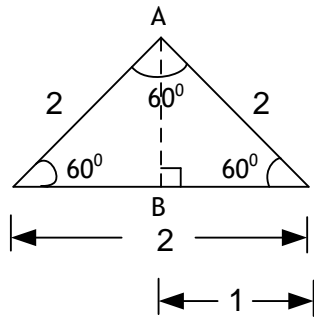
$$AB = \sqrt{2^2 - 1} = \sqrt{3}$$

$$\sin 60^0 = \frac{\sqrt{3}}{2}$$

Ex-17-4: Fig-17-4 shows a curve which passes through the points shown in the following table.

Fig-17-4
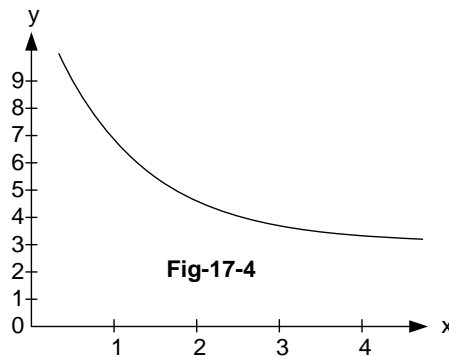
x	1	1.5	2	2.5	3	3.5	4
y	8.2	6.4	5.5	5	4.7	4.4	4.2

Use the trapezium rule with 6 strips to estimate the area of the region bounded by the curve, the lines x=1 and x=4, and the x-axis.
State, with a reason, whether the trapezium rule gives an overestimate or an underestimate of the area of this region.

Ans: $Area \approx \dfrac{h}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + y_6\right)$

$h = \dfrac{b-a}{n} = \dfrac{4-1}{6} = 0.5$

$Area \approx \dfrac{0.5}{2}\left[8.2 + 2(6.4 + 5.5 + 5 + 4.7 + 4.4) + 4.2\right] = 0.25(64.4) = 16.1$

$\Rightarrow \displaystyle\int_0^{10} \dfrac{1}{4}\left(10x - x^2\right)dx \approx \dfrac{2}{2}\left[0 + 2(5.64 + 6.44 + 6.44 + 5.64) + 0\right] = 48.32\, m^2$

Hence, overestimated

Ex-17-5: (i) Sketch the graph of by $y = \tan x$ for $0^0 \le x \le 360^0$.
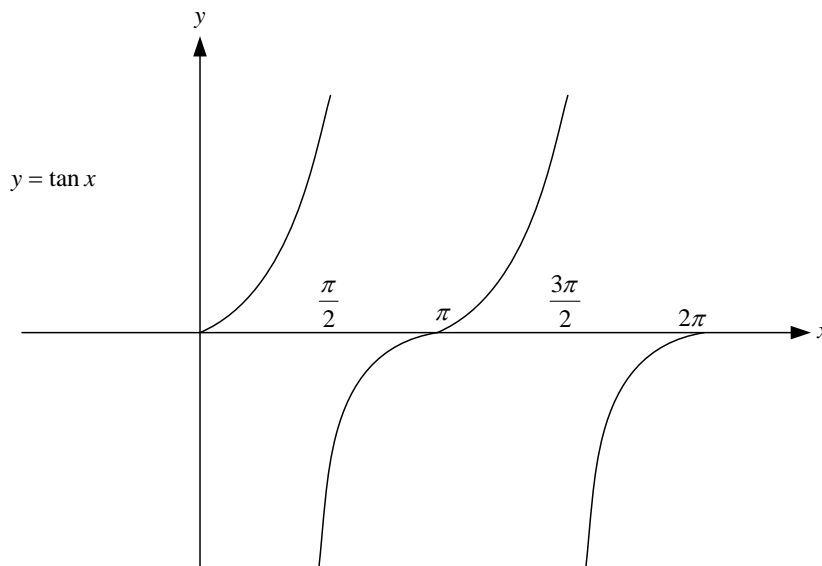
(iii) Solve the equation by $4\sin x = 3\cos x$ for $0^0 \le x \le 360^0$.

Ans: $4\sin x = 3\cos x \quad \Rightarrow \dfrac{\sin x}{\cos x} = \tan x = \dfrac{3}{4} \quad \Rightarrow x = 36.87^0,\ 216.87^0$

Ex-17-6: A curve has gradient given by $\dfrac{dy}{dx} = x^2 - 6x + 9$. Find $\dfrac{d^2 y}{dx^2}$.

Ans: $\dfrac{dy}{dx} = x^2 - 6x + 9 \quad \Rightarrow \dfrac{d^2 y}{dx^2} = 2x - 6x$

Show that the curve has a stationary point of inflection when x = 3.
Ans: For stationary point the following should apply.

$$\frac{dy}{dx} = gradient = 0 \quad and \quad \frac{d^2y}{dx^2} < 0 \quad for \quad Maximum$$

$$\frac{dy}{dx} = gradient = 0 \quad and \quad \frac{d^2y}{dx^2} > 0 \quad for \quad Minimum$$

$$\frac{dy}{dx} = x^2 - 6x + 9 = 0 \quad \Rightarrow (x-3)^2 = 0 \quad \Rightarrow x = 3$$

Hence, there is a stationary point at x=3

Ex-17-7: In Fig-17-7, A and B are points on the circumference of a circle with centre O.

Angle AOB = 1.2 radians.
The arc length AB is 6 cm.
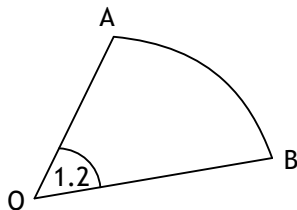
Fig-17-7

(i)   Calculate the radius of the circle.

(ii)   Calculate the length of the cord AB.
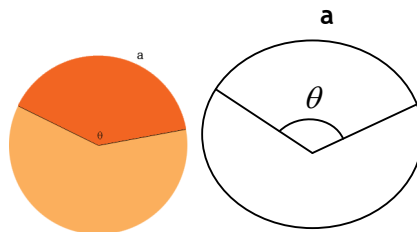
And the length of the arc is **a = r θ** $\Rightarrow a = r\theta$ $\Rightarrow r = \dfrac{a}{\theta} = \dfrac{6}{1.2} = 5\,cm$
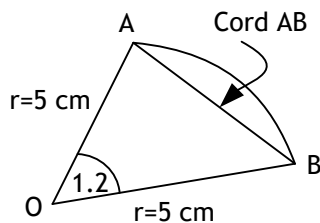
$$1.2\,radians = \frac{1.2 \times 180}{\pi} = 68.755^0$$

$$AB^2 = 5^2 + 5^2 - 2(5)(5)\cos 68.755^0 = 50 - 50\cos 68.755^0 = 31.88 \implies AB = \sqrt{31.88} = 5.646\,cm$$

Ex-17-8: Find $\int\left(x^{\frac{1}{2}} + \dfrac{6}{x^3}\right)dx = \int\left(x^{\frac{1}{2}} + 6x^{-3}\right)dx = \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + \dfrac{6x^{-2}}{-2} = \dfrac{3}{2}x^{\frac{3}{2}} - \dfrac{3}{x^2} + C$

Ex-17-9: The graph of $\log_{10} y$ against x is a straight line as shown in Fig-17-9.

Fig-17-9
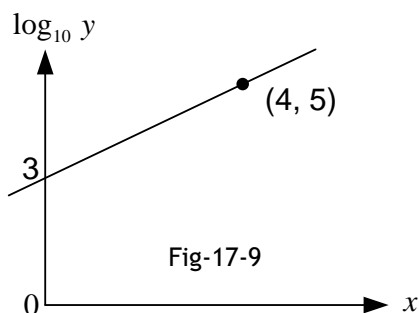
(i)   Find the equation for $\log_{10} y$ in terms of x.

Ans: $\log_{10} y = mx + c \qquad m = \dfrac{5-3}{4-0} = \dfrac{2}{4} = \dfrac{1}{2}$

$\implies \log_{10} y = \dfrac{1}{2}x + c \implies 3 = \dfrac{1}{2}(0) + c \implies c = 3$

$\implies \log_{10} y = \dfrac{1}{2}x + 3$

(ii)   Find the equation for y in terms of x.

$\implies \log_{10} y = \dfrac{1}{2}x + 3 \implies y = 10^{\left(\frac{1}{2}x+3\right)}$

SECTION B (P-17)

Ex-17-10: The equation of a curve is $y = 7 + 6x - x^2$

(i)   Use calculus to find the coordinates of the turning point on this curve.

Find also the coordinates of the points of intersection of this curve with the axes, and sketch the curve.

(ii)   Find $\int_{1}^{5}(7+6x-x^2)dx$, showing your working.

(iii)   The curve and the line y = 12 intersect at (1, 12) and (5,12). Using your answer to part (ii), find the area of the finite region between the curve and the line y = 12.

Ans: For stationary point the following should apply.

$$\frac{dy}{dx}=gradient=0 \quad and \quad \frac{d^2y}{dx^2}<0 \quad for \ Maximum$$

$$\frac{dy}{dx}=gradient=0 \quad and \quad \frac{d^2y}{dx^2}>0 \quad for \ Minimum$$

$$y=7+6x-x^2 \ \Rightarrow\frac{dy}{dx}=6-2x=0 \ \Rightarrow x=3 \ \ \Rightarrow Po\,int=(3,16)$$

This curve intersects x-axis when y = 0;

$$y=7+6x-x^2=0 \ \Rightarrow x^2-6x-7=0 \ \Rightarrow(x-7)(x+1)=0 \ \Rightarrow x=7, \ and \ x=-1$$
$$\Rightarrow P_1=(-1,0) \ \ and \ P_2=(7,0)$$

This curve intersects y-axis when x = 0;

$$y=7+6x-x^2=7 \ \Rightarrow P_1=(0,7)$$

Ans:

$$\int_{1}^{5}(7+6x-x^2)dx=7x+\frac{6x^2}{2}-\frac{x^3}{3}\Big|_{1}^{5}=7(5)+\frac{6(5)^2}{2}-\frac{5^3}{3}-\left(7(1)+\frac{6(1)^2}{2}-\frac{1^3}{3}\right)=35+75-\frac{125}{3}-7-3+\frac{1}{3}$$
$$=110.333-51.667=58.667$$
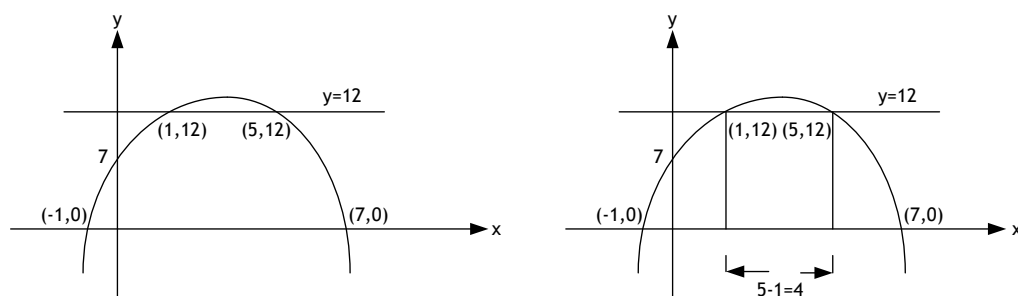
Ans: $Area=\int_{1}^{5}(7+6x-x^2)dx-\left(4\times12\right)=58.667-48=10.667 \ unit^2$

Ex-17-11: The equation of the curve shown in Fig-17-11 is $y = x^3 - 6x + 2$.

Fig-17-11
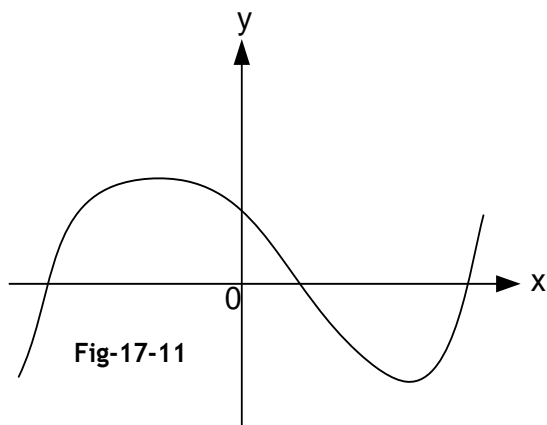
(i) Find $\dfrac{dy}{dx}$

$y = x^3 - 6x + 2 \quad \Rightarrow \dfrac{dy}{dx} = 3x^2 - 6$

(ii) Find, in exact form, the range of values of x for which $x^3 - 6x + 2$ is a decreasing function.

Ans: The function is decreasing when

$\dfrac{dy}{dx} = 3x^2 - 6 < 0 \quad \Rightarrow x^2 < 2 \quad \Rightarrow -\sqrt{2} \le x \le \sqrt{2}$.

(iii) Find the equation of the tangent to the curve at the point (-1, 7).

Find also the coordinates of the point where this tangent crosses the curve again.

Ans: Gradient of the tangent$= m = \dfrac{dy}{dx} = 3x^2 - 6 \Big|_{x=-1} = 3 - 6 = -3$.

Ans: $y = -3x + c \quad \Rightarrow 7 = -3(-1) + c \quad \Rightarrow c = 7 - 3 = 4$

$\Rightarrow y = -3x + 4$

If this tangent crosses the curve again, then

$x^3 - 6x + 2 = -3x + 4 \quad \Rightarrow x^3 - 3x - 2 = 0 \quad \Rightarrow x = -1 \ and \ x = 2 \ are the solution$

The coordinates of the other point is: (2, -2)

Ex-17-12:   (i)   Granny gives Simon £5 on his 1st birthday. On each successive birthday,

she gives him £2 more than she did the previous year.

(A)   How much does she give him on his 10th birthday?

Ans: This is arithmetic progression or series. The nth term of an arithmetic sequence is given by:

$$a_n = a + (n-1)d = 5 + 2(10-1) = 5 + 18 = 23$$

(B)   How old is he when she gives him £51?

Ans: $a_n = a + (n-1)d \quad \Rightarrow 51 = 5 + 2(n-1) \quad \Rightarrow 2n = 51 - 3 = 48 \quad \Rightarrow n = 24$

(C)   How much has she given him in total when he has had his 20th birthday present?

Ans: $S_n = \dfrac{n}{2}[2a + (n-1)d] = \dfrac{20}{2}[2 \times 5 + 2(20-1)] = 10[10 + 38] = 480$

(ii)   Grandpa gives Simon £5 on his 1st birthday and increases the amount by 10% each year.

(A)   How much does he give Simon on his 10th birthday?

Ans: $a_n = ar^{n-1} = 5(1.1)^{10-1} = 5(1.1)^9 = 5(2.3579) = 11.79 \quad because \ r = 1.1$

This is a geometric series and r=1.1

(B)   Simon first gets a present of over £50 from Grandpa on his nth birthday. Show that $n > \dfrac{1}{\log_{10} 1.1} + 1$ Find the value of n.

Ans: $a_n = ar^{n-1} \quad \Rightarrow 50 = 5(1.1)^{n-1} \quad \Rightarrow \log_{10} 50 = \log_{10} 5(1.1)^{n-1}$

$$\log_{10} 50 = \log_{10}(5 \times 10) = \log_{10} 5 + \log_{10} 10 = \log_{10} 5 + 1$$

$$and \quad \log_{10} 5(1.1)^{n-1} = \log_{10} 5 + \log_{10}(1.1)^{n-1} = \log_{10} 5 + (n-1)\log_{10} 1.1$$

$$\Rightarrow \log_{10} 5 + 1 = \log_{10} 5 + (n-1)\log_{10} 1.1 \Rightarrow n - 1 = \dfrac{1}{\log_{10} 1.1} \quad \Rightarrow n = \dfrac{1}{\log_{10} 1.1} + 1 = 26$$

PAPER-18

Ex-18-1: $\displaystyle\sum_{r=3}^{6} \frac{12}{r}$

Ans: $\displaystyle\sum_{r=3}^{6} \frac{12}{r} = \frac{12}{3} + \frac{12}{4} + \frac{12}{5} + \frac{12}{6} = 4 + 3 + 2.4 + 2 = 11.4$

OR $\displaystyle\sum_{r=3}^{6} \frac{12}{r} = \frac{12}{3} + \frac{12}{4} + \frac{12}{5} + \frac{12}{6} = 4 + 3 + 2\frac{2}{5} + 2 = 11\frac{2}{5}$

Ex-18-2: $\displaystyle\int \left(3x^5 + 2x^{-\frac{1}{2}} \right) dx$

Ans: $\displaystyle\int \left(3x^5 + 2x^{-\frac{1}{2}} \right) dx = \frac{3x^6}{6} + 2\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{x^6}{2} + 4x^{\frac{1}{2}} + c$

Ex-18-3: At a place where a river is 7.5 m wide, its depth is measured every 1.5 m across the river. The table shows the result.

Distance across river (m)	0	1.5	3	4.5	6	7.5
Depth of river (m)	0.6	2.3	3.1	2.8	1.8	0.7

Use the trapezium rule with 5 stripes to estimate the area of cross-section of the river.

Ans:

$$Area \approx \frac{h}{2}\left(y_0 + y_5 + 2y_1 + 2y_2 + 2y_3 + 2y_4\right)$$

$$h = \frac{b-a}{n} = \frac{7.5-0}{5} = 1.5$$

$$Area \approx \frac{1.5}{2}\left[0.6 + 0.7 + 2(2.3 + 3.1 + 2.8 + 1.8)\right] = 0.75(21.3) = 15.975 \cong 16.0 \ (3\,s.f)$$

Ans: Look at The following first, before answering this question.

$y = f(x) = x^2 + x - 12$

$Minimum\ point = \left(-\dfrac{1}{2}, -12\dfrac{1}{4}\right)$

$y = f(x) = x^2 + x - 12 = (x+4)(x-3) = 0$

$\Rightarrow x = -4,\ x = 3 \Rightarrow point\,s\ are:\ (-4,0)\ and\ (3,0)$

$y = f(x) = x^2 + x - 12$

$\dfrac{dy}{dx} = 2x + 1 = 0 \Rightarrow x = -\dfrac{1}{2}$

$y\Big|_{x=-\frac{1}{2}} = \left(-\dfrac{1}{2}\right)^2 - \left(\dfrac{1}{2}\right) - 12 = \dfrac{1}{4} - \dfrac{1}{2} - 12 = -12\dfrac{1}{4}$

$\Rightarrow Minimum\ point = \left(-\dfrac{1}{2}, -12\dfrac{1}{4}\right)$

$y_1 = 3f(x) = 3x^2 + 3x - 36$

$Minimum\ point = \left(-\dfrac{1}{2}, -36\dfrac{3}{4}\right)$

$y_1 = 3f(x) = 3x^2 + 3x - 36 = 3(x+4)(x-3) = 0$

$\Rightarrow x = -4,\ x = 3 \Rightarrow point\,s\ are:\ (-4,0)\ and\ (3,0)$

$y_1 = 3f(x) = 3x^2 + 3x - 36$

$\dfrac{dy_1}{dx} = 6x + 3 = 0 \Rightarrow x = -\dfrac{1}{2}$

$y\Big|_{x=-\frac{1}{2}} = 3\left(-\dfrac{1}{2}\right)^2 - 3\left(\dfrac{1}{2}\right) - 36 = \dfrac{3}{4} - \dfrac{3}{2} - 36 = -36\dfrac{3}{4}$

$\Rightarrow Minimum\ point = \left(-\dfrac{1}{2}, -36\dfrac{3}{4}\right)$
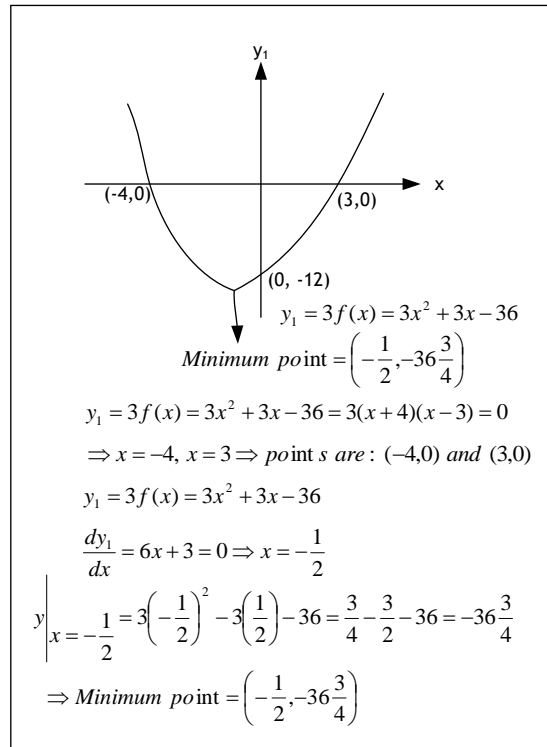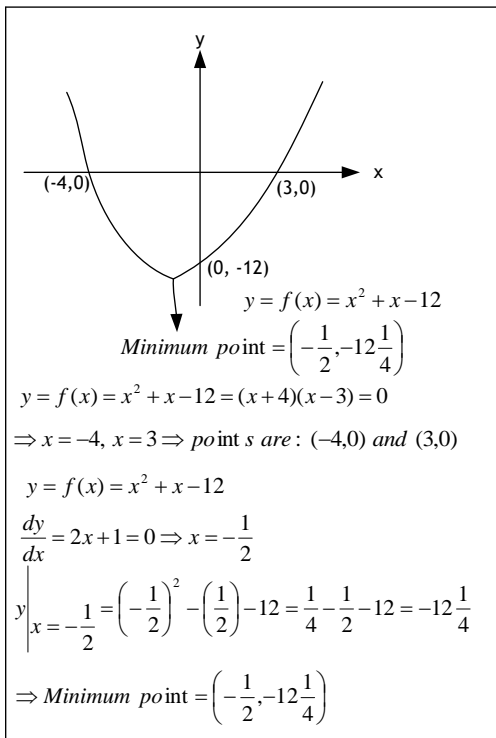
Note: as it can be seen very clearly, that the y-value of the minimum/maximum point is affected. If $y = f(x)$ has a minimum/maximum point of $P_1(a,b)$, then $y = 3f(x)$ will have a minimum/maximum point of $P_2(a,3b)$. For example, if $y = f(x)$ has the minimum point of $P_1\left(-\dfrac{1}{2}, -12\dfrac{1}{4}\right)$, then the minimum/maximum point for $y = 3f(x)$ will be $P_2\left(-\dfrac{1}{2}, -36\dfrac{3}{4}\right)$.

Also: (very important): If $y = f(x)$ has a minimum/maximum point of $P_1(a,b)$, then $y = -3f(x)$ will have a maximum/minimum point of $P_2(a,-3b)$. For example, if $y = f(x)$ has the minimum point of $P_1\left(-\dfrac{1}{2}, -12\dfrac{1}{4}\right)$, then the maximum point for $y = -3f(x)$ will be $P_2\left(-\dfrac{1}{2}, 36\dfrac{3}{4}\right)$.

Look at this also:

$$y = f(x) = x^2 + x - 12$$

$$Minimum\ point = \left(-\frac{1}{2}, -12\frac{1}{4}\right)$$

$$y = f(x) = x^2 + x - 12 = (x+4)(x-3) = 0$$

$$\Rightarrow x = -4,\ x = 3 \Rightarrow points\ are:\ (-4,0)\ and\ (3,0)$$

$$y = f(x) = x^2 + x - 12$$

$$\frac{dy}{dx} = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$$

$$y\Big|_{x=-\frac{1}{2}} = \left(-\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 12 = \frac{1}{4} - \frac{1}{2} - 12 = -12\frac{1}{4}$$

$$\Rightarrow Minimum\ point = \left(-\frac{1}{2}, -12\frac{1}{4}\right)$$

$$Max\_point = \left(-\frac{1}{2}, 36\frac{3}{4}\right)$$

$$y_1 = -3f(x) = -3x^2 - 3x + 36$$

$$y_1 = -3f(x) = -3x^2 - 3x + 36 = -3(x+4)(x-3) = 0$$

$$\Rightarrow x = -4,\ x = 3 \Rightarrow points\ are:\ (-4,0)\ and\ (3,0)$$

$$y_1 = -3f(x) = -3x^2 - 3x + 36$$

$$\frac{dy_1}{dx} = -6x - 3 = 0 \Rightarrow x = -\frac{1}{2}$$

$$y\Big|_{x=-\frac{1}{2}} = -3\left(-\frac{1}{2}\right)^2 - 3\left(-\frac{1}{2}\right) + 36 = -\frac{3}{4} + \frac{3}{2} + 36 = 36\frac{3}{4}$$

$$\Rightarrow Max\ point = \left(-\frac{1}{2}, 36\frac{3}{4}\right)$$

Also look here:

$$y = f(x) = x^2 + x - 12$$

$$Minimum\ point = \left(-\frac{1}{2}, -12\frac{1}{4}\right)$$

$$y = f(x) = x^2 + x - 12 = (x+4)(x-3) = 0$$

$$\Rightarrow x = -4,\ x = 3 \Rightarrow points\ are:\ (-4,0)\ and\ (3,0)$$

$$y = f(x) = x^2 + x - 12$$

$$\frac{dy}{dx} = 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$$

$$y\Big|_{x=-\frac{1}{2}} = \left(-\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) - 12 = \frac{1}{4} - \frac{1}{2} - 12 = -12\frac{1}{4}$$

$$\Rightarrow Minimum\ point = \left(-\frac{1}{2}, -12\frac{1}{4}\right)$$

$$y_1 = f(2x) = (2x)^2 + 3(2x) - 12$$

$$Minimum\ point = \left(-\frac{1}{4}, -12\frac{1}{4}\right)$$

$$y_1 = f(2x) = (2x)^2 + (2x) - 12 = 4x^2 + 2x - 12 = 0$$

$$y_1 = 4x^2 + 2x - 12 = 2x^2 + x - 6 = 0 \Rightarrow (2x-3)(x+2) = 0$$

$$\Rightarrow x = -2,\ x = 1.5 \Rightarrow points\ are:\ (-2,0)\ and\ (1.5,0)$$

$$y_1 = 4x^2 + 2x - 12 = 0$$

$$\frac{dy_1}{dx} = 8x + 2 = 0 \Rightarrow x = -\frac{1}{4}$$

$$y\Big|_{x=-\frac{1}{4}} = 4\left(-\frac{1}{4}\right)^2 + 2\left(-\frac{1}{4}\right) - 12 = \frac{1}{4} - \frac{1}{2} - 12 = -12\frac{1}{4}$$

$$\Rightarrow Minimum\ point = \left(-\frac{1}{4}, -12\frac{1}{4}\right)$$

Now lets repeat the question again:

Ex-18-4:   The curve $y = f(x)$ has a minimum point at $(3, 5)$.

State the coordinates of the corresponding minimum point on the graph of

(i) $y = 3f(x)$

(ii) $y = f(2x)$

Ans:

The curve $y = f(x)$ has a minimum point at $(3, 5)$.

(i) $y = 3f(x)$ has a minimum point at $(3, 3\times 5) = (3, 15)$.

(ii) $y = f(2x)$ has a minimum point at $\left(\dfrac{1}{2}\times 3, 5\right) = \left(\dfrac{3}{2}, 5\right)$.

Ex-18-5:   The second term of a geometric sequence is 6 and the fifth term is -48.

Find the tenth term of the sequence.

Find also, in simplified form, an expression for the sum of the first n terms of this sequence.

Ans:

$$u_k = ar^{k-1}$$

$$u_2 = ar^{2-1} = ar = 6 \qquad ....(1)$$

$$u_5 = ar^{5-1} = ar^4 = -48 \qquad ....(2)$$

Dividing Eq.2 by 1:

$$\frac{ar^4}{ar} = \frac{-48}{6} \Rightarrow r^3 = -8 \Rightarrow r = -2$$

$$ar = 6 \Rightarrow a = \frac{6}{r} = \frac{6}{-2} = -3$$

$$u_k = ar^{k-1} \Rightarrow u_{10} = ar^{10-1} = ar^9 = -3\times(-2)^9 = ?$$

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{-3((-2)^n - 1)}{-2 - 1} = \frac{-3((-2)^n - 1)}{-3} = (-2)^n - 1$$

Ex-18-6: The third term of an arithmetic progression is 24. The tenth term is 3.

Find the first term and the common difference.

Find also, the sum of the 21st to 50th terms inclusive

Ans:

$$u_n = a + (n-1)d$$

$$u_3 = a + (3-1)d = a + 2d = 24 \quad ....(1)$$

$$u_{10} = a + (10-1)d = a + 9d = 3 \quad ....(2)$$

Subtracting Eq.2 from 1:

$$-7d = 21 \Rightarrow d = -3$$

$$a = 24 - 2d = 24 - 2(-3) = 30$$

$$S_n = \frac{1}{2}n[2a + (n-1)d] = \frac{1}{2} \times 30[2 \times 30 + (30-1) \times -3] = 15[60 - 87] = -405$$

Ex-18-7: Simplify

(i) $\log_{10} x^5 + 3\log_{10} x^4$

(ii) $\log_{10} 1 - \log_a a^b$

Ans:

(i) $\log_{10} x^5 + 3\log_{10} x^4 = 5\log_{10} x + 12\log_{10} x = 17\log_{10} x = \log_{10} x^{17}$

(ii) $\log_a 1 - \log_a a^b = 0 - b\log_a a = -b$

Ex-18-8: Showing your method clearly, solve the equation

$$5\sin^2 \theta = 5 + \cos\theta \qquad \text{for } 0^0 \le \theta \le 360^0$$

Ans:

$$5\sin^2 \theta = 5 + \cos\theta$$

Remember: $\sin^2 \theta + \cos^2 \theta = 1 \Rightarrow \sin^2 \theta = 1 - \cos^2 \theta$

$$\Rightarrow 5\sin^2 \theta = 5(1 - \cos^2 \theta) = 5 + \cos\theta \Rightarrow 5 - 5\cos^2 \theta = 5 + \cos\theta$$

$$\Rightarrow -5\cos^2\theta - \cos\theta = 0 \Rightarrow \cos\theta(5\cos\theta + 1) = 0$$

$$\Rightarrow \cos\theta(5\cos\theta + 1) = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90^0,\ 270^0$$

$$If\ \ 5\cos\theta + 1 = 0 \Rightarrow \cos\theta = -\frac{1}{5} \Rightarrow \theta = 101.54^0,\ 258^0$$

Ans: $\theta = 101.54^0,\ 90^0, 270^0,\ 258^0$

Ex-18-9: Charles has a slice of cake; its cross-section is a sector of a circle, as shown in Fig-18-9. The radius is r cm and the sector angle is $\frac{\pi}{6}$ radians.

He wants to give half of the slice to Jan. He makes a cut across the sector as shown.

Fig-18-9
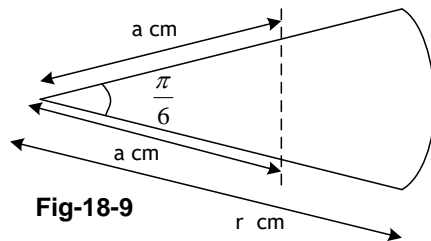
Show that when they each have half the slice, $a = \sqrt{\dfrac{\pi}{6}}$

Ans: Remember this:
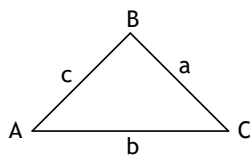
$$Area = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$$

Area of the total sector$= \dfrac{1}{2}r^2\theta = \dfrac{1}{2}r^2 \times \dfrac{\pi}{6} = \dfrac{\pi r^2}{12}$

Area of the triangle$= \dfrac{Sector - Area}{2} = \dfrac{1}{2}\left(\dfrac{\pi r^2}{12}\right) = \dfrac{\pi r^2}{24}$

Also Area of the triangle$= = \dfrac{1}{2} \times a \times a \times \sin\dfrac{\pi}{6} = \dfrac{1}{2}a^2\sin\dfrac{\pi}{6} = \dfrac{a^2}{4}$

$$\Rightarrow \frac{a^2}{4} = \frac{\pi r^2}{24} \Rightarrow a^2 = \frac{\pi r^2}{6} \Rightarrow a = \sqrt{\frac{\pi r^2}{6}} = r\sqrt{\frac{\pi}{6}}$$

Ex-18-10: A is the point with coordinates (1, 4) on the curve $y = 4x^2$. B is the point with coordinates (0, 1) as shown in Fig-18-10.

Fig-18-10
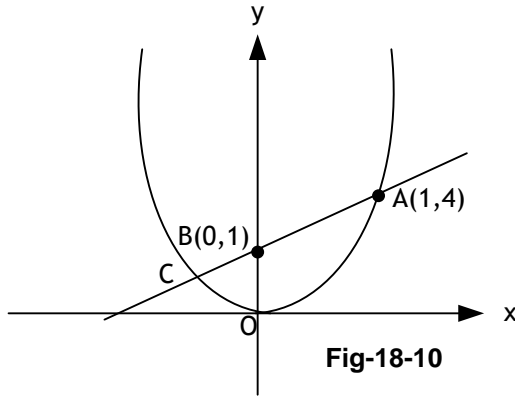
(i) The line through A and B intersects the curve again at the point C. Show that the coordinates of C are $\left(-\dfrac{1}{4}, \dfrac{1}{4}\right)$.

(ii) Use calculus to find the equation of the tangent to the curve at A and verify that equation of the tangent at C is $y = -2x - \dfrac{1}{4}$.

(iii) The two tangents intersect at the point D. Find the y-coordinate of D.

Ans:
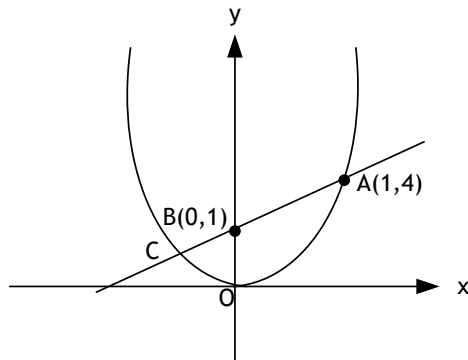
$Line\ AB:\quad y = mx + c \Rightarrow m = \dfrac{4-1}{1-0} = 3 \Rightarrow y = 3x + c \Rightarrow 1 = 0 + c \Rightarrow y = 3x + 1$

Now $y = 3x + 1$ and $y = 4x^2$ should intersect at C.

$\Rightarrow 4x^2 = 3x + 1 \Rightarrow 4x^2 - 3x - 1 = 0 \Rightarrow (4x+1)(x-1) = 0 \Rightarrow x = -1\ and\ x = -\dfrac{1}{4}$

$$y = 4x^2 = 4\left(-\frac{1}{4}\right)^2 = \frac{4}{16} = \frac{1}{4} \Rightarrow C\left(-\frac{1}{4}, \frac{1}{4}\right)$$

$$y = 4x^2$$

$$y = mx + c$$

$$m = \frac{dy}{dx} = 8x\Big|_{x=1} = 8 \Rightarrow y = 8x + c \Rightarrow 4 = 8(1) + c \Rightarrow c = -4 \Rightarrow y = 8x - 4$$

$$m = \frac{dy}{dx} = 8x\Big|_{x=-\frac{1}{4}} = -2 \Rightarrow y = -2x + c \Rightarrow \frac{1}{4} = -2\left(-\frac{1}{4}\right) + c \Rightarrow c = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \Rightarrow y = -2x - \frac{1}{4}$$

$$y = -2x - \frac{1}{4} \quad \text{and} \quad y = 8x - 4 \quad \text{intersect}$$

$$\Rightarrow -2x - \frac{1}{4} = 8x - 4 \Rightarrow -10x = -4 + \frac{1}{4} = -\frac{15}{4} \Rightarrow x = \frac{15}{40} = \frac{3}{8} \quad and \quad y = 8x - 4 = 8\left(\frac{3}{8}\right) - 4 = -1$$

Intersection point is: $\left(\frac{3}{8}, -1\right)$

Ex-18-11: Fig-18-11 shows the curve $y = x^3 - 3x^2 - x + 3$.

(i) Use calculus to find $\int_1^3 (x^3 - 3x^2 - x + 3)dx$ and state what this represents.

(ii) Find the x-coordinates of the turning points of the curve $y = x^3 - 3x^2 - x + 3$, giving your answers in surds form. Hence, state the set of values of x for which $y = x^3 - 3x^2 - x + 3$, is a decreasing function.

Fig-18-11
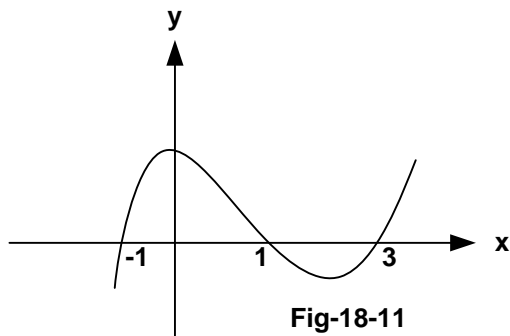
Ans:

$$\int_1^3 \left(x^3 - 3x^2 - x + 3\right)dx = \frac{x^4}{4} - \frac{3x^3}{3} - \frac{x^2}{2} + 3x = \frac{x^4}{4} - x^3 - \frac{x^2}{2} + 3x \Big|_1^3$$

$$\int_1^3 \left(x^3 - 3x^2 - x + 3\right)dx = \frac{3^4}{4} - 3^3 - \frac{3^2}{2} + 3\times 3 - \left(\frac{1^4}{4} - 1^3 - \frac{1^2}{2} + 3\right) = \frac{81}{4} - 27 - \frac{9}{2} + 9 - \left(\frac{1}{4} - 1 - \frac{1}{2} + 3\right) =$$

The above integral represents area under the curve.

(b)

$$y = x^3 - 3x^2 - x + 3$$

$$\frac{dy}{dx} = 3x^2 - 6x - 1 = 0 \Rightarrow x_{1,2} = \frac{6 \pm \sqrt{36 + 12}}{6} = \frac{6 \pm \sqrt{48}}{6} = \frac{6 \pm 4\sqrt{3}}{6} = \frac{3 \pm 2\sqrt{3}}{3}$$

$$\Rightarrow x_1 = \frac{3 + 2\sqrt{3}}{3} \quad and \quad x_2 = \frac{3 - 2\sqrt{3}}{3}$$

$$\frac{dy}{dx} = 3x^2 - 6x - 1 > 0 \quad for\ increasin\,g\ function \quad \textbf{Try This!}$$

Ex-18-12: The table shows the size of a population of house sparrow from 1980 to 2005.

Year (t)	1980	1985	1990	1995	2000	2005
Population (P)	25000	22000	18750	16250	13500	12000

The 'red alert' category for birds is used when a population has decreased by at least 50% in the previous 25 years.

(i)   Show that the information for this population is consistent with the house sparrow being on red alert in 2005.

The size of the population may be modelled by a function of the form $P = a \times 10^{-kt}$, where P is the population, t is the the number of years after 1980, and **a** and **k** are constants.

(ii)   Write the equation $P = a \times 10^{-kt}$ in logarithmic form using base 10, giving your answer as simply as possible.

(iii)   Complete the table and draw the graph of $\log_{10} P$ against t, drawing a line of best fit by eye.

(iv)   Use your graph to find the values of **a** and **k** and hence the equation for **P** in terms of **t**.

(v)   Find the size of the population in 2015 as predicted by this model

Would the house sparrow still be on red alert? Give a reason for your answer.

Ans:

$P = a \times 10^{-kt}$

$\log P = \log(a \times 10^{-kt}) = \log a - kt \log 10 = -kt + \log a$

$\log P = -kt + \log a$

$y = mx + c$

Year (t)	1980	1985	1990	1995	2000	2005
Population (P)	25000	22000	18750	16250	13500	12000
$\log_{10} P$	4.4	4.34	4.27	4.2	4.1	4.08
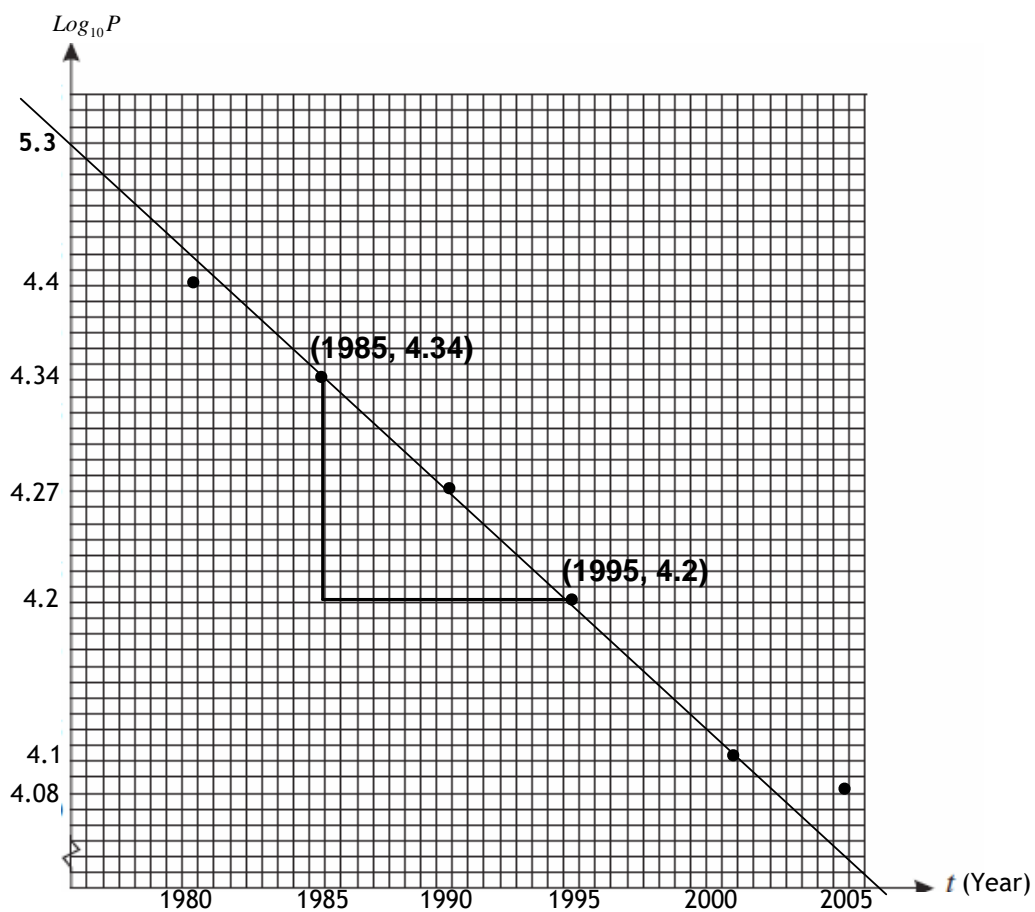
$$Gradient = \frac{4.2 - 4.34}{1995 - 1985} = -\frac{0.14}{10} = -0.014$$

$$c = y - \text{int } ercept = 5.3$$

$$\log P = -kt + \log a$$

$$\log a = c = y - \text{int } ercept = 5.3 \Rightarrow a \cong 200000$$

$$- k = gradient = -0.014 \Rightarrow k = 0.014$$

$$y = mx + c$$

$$\log_{10} P = -0.014t + 5.3$$

$$\log_{10} P = -0.014 \times 2015 + 5.3 = -28.21 + 5.3 = -22.91$$

$$P = a \times 10^{-kt} = 200000 \times 10^{-0.014t} \Big|_{t = 2015} = 39810 \times 10^{-0.015 \times 2014} = ?$$

PAPER-19

SECTION A (C2-27-5-10)

Ex-19-1: You are given that $u_1 = 1$

$$u_{n+1} = \frac{u_n}{1 + u_n}$$

Find the values of u_2, u_3 and u_4. Give your answers as fractions.

$$u_2 = \frac{u_1}{1 + u_1} = \frac{1}{1 + 1} = \frac{1}{2}$$

$$u_3 = \frac{u_2}{1 + u_2} = \frac{0.5}{1 + 0.5} = \frac{1}{3}$$

$$u_4 = \frac{u_3}{1 + u_3} = \frac{\frac{1}{3}}{1 + \frac{1}{3}} = \frac{1}{4}$$

Ex-19-2: (i)

$$\sum_{r=2}^{5} \frac{1}{r-1} = \frac{1}{2-1} + \frac{1}{3-1} + \frac{1}{4-1} + \frac{1}{5-1} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{12+6+4+3}{12} = \frac{25}{12}$$

(ii) Express the series $2\times 3 + 3\times 4 + 4\times 5 + 5\times 6 + 6\times 7$ in the form $\sum_{r=2}^{a} f(r)$

where $f(r)$ and a are to be determined.

Ans: $2\times 3 + 3\times 4 + 4\times 5 + 5\times 6 + 6\times 7 = \sum_{r=2}^{6} r(r+1)$

Ex-19-3: (i) Differentiate $x^3 - 6x^2 - 15x + 50$

$$y = x^3 - 6x^2 - 15x + 50$$

$$\frac{dy}{dx} = 3x^2 - 12x - 15$$

(ii) Hence find the x - coordinates of the stationary points on the curve
$y = x^3 - 6x^2 - 15x + 50$

$$\frac{dy}{dx} = 3x^2 - 12x - 15 = 0 \quad \Rightarrow (3x+3)(x-5) = 0 \quad \Rightarrow x = 5 \quad and \quad \Rightarrow x = -1$$

$$y\Big|_{x=5} = x^3 - 6x^2 - 15x + 50\Big|_{x=5} = 5^3 - 6(5)^2 - 15(5) + 50 = 125 - 150 - 75 + 50 = -50$$

$$y\Big|_{x=-1} = x^3 - 6x^2 - 15x + 50\Big|_{x=-1} = (-1)^3 - 6(-1)^2 - 15(-1) + 50 = -1 - 6 + 15 + 50 = 58$$

Coordinates: $P_1(5,-50)$ and $P_2(-1,58)$

$$\frac{dy}{dx} = 3x^2 - 12x - 15 = 0 \quad \Rightarrow \frac{d^2y}{dx^2} = 6x - 12\Big|_{x=5} = 30 - 12 = 18 \quad \Rightarrow P_1(5,-50) \; Minimum$$

$$and \quad \frac{d^2y}{dx^2} = 6x - 12\Big|_{x=-1} = -6 - 12 = -18 \quad \Rightarrow P_2(-1,58) \; Max$$

Ex-19-4: In this question, $f(x) = x^2 - 5x$. Fig-19-4 shows a sketch of the graph of $y = f(x)$.

y

Fig-19-4

0   5   x

(2.5, -6.25)
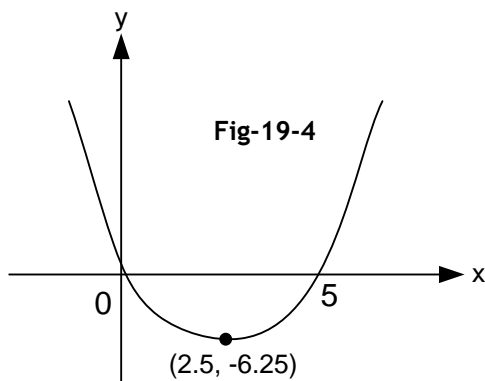
On separate diagrams, sketch the curves $y = f(2x)$ and $y = 3f(x)$, labelling the coordinates of their intersections with the axes and their turning points.

y

f(x)

Fig.4

0   5   x

(2.5, -6.25)

y

f(2x)

2.5

0   5   x

(1.25, -6.25)

y

3f(x)

0   5   x

(2.5, -18.75)
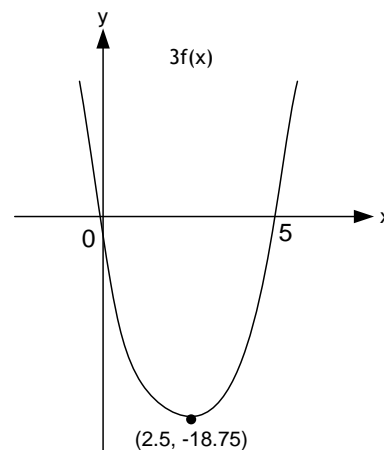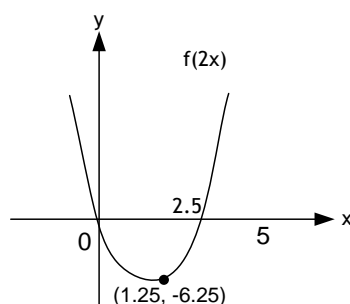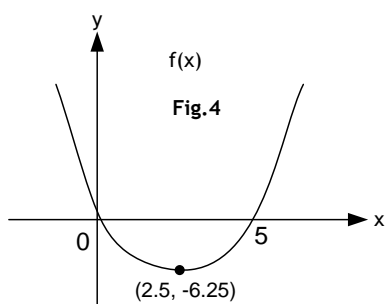
Ex-19-5: Find $\displaystyle\int_{2}^{5}\left(1 - \frac{6}{x^3}\right)dx$

$$\int_{2}^{5}\left(1 - \frac{6}{x^3}\right)dx = \int_{2}^{5}\left(1 - 6x^{-3}\right)dx = x - \frac{6}{-2}x^{-2} = x + \frac{3}{x^2}\bigg|_{2}^{5} = 5 + \frac{3}{25} - \left(2 + \frac{3}{4}\right) = 5 - 2 + \frac{3}{25} - \frac{3}{4} = \frac{237}{100} = 2.37$$

Ex-19-6: The gradient of a curve is $6x^2 + 12x^{\frac{1}{2}}$. The curve passes through the point (4, 10). Find the equation of the curve.

$$\frac{dy}{dx} = 6x^2 + 12x^{\frac{1}{2}} \Rightarrow y = \int\left(6x^2 + 12x^{\frac{1}{2}}\right)dx = \frac{6x^3}{3} + 12\left(\frac{2}{3}\right)x^{\frac{3}{2}} + c$$

$$y = \frac{6x^3}{3} + 12\left(\frac{2}{3}\right)x^{\frac{3}{2}} + c = 2x^3 + 8x^{\frac{3}{2}} + c \Rightarrow 10 = 128 + 64 + c \quad \Rightarrow c = 10 - 192 = -182$$

$$\Rightarrow y = 2x^3 + 8x^{\frac{3}{2}} - 182$$

Ex-19-7: Express $\log_a x^3 + \log_a \sqrt{x}$ in the form $k\log_a x$.

Ans:

$$\log_a x^3 + \log_a \sqrt{x} = 3\log_a x + \log_a (x)^{\frac{1}{2}} = 3\log_a x + \frac{1}{2}\log_a x = \frac{7}{2}\log_a x$$

Ex-19-8: Showing your method clearly, solve the equation $4\sin^2\theta = 3 + \cos^2\theta$, for values of θ between 0^0 and 360^0.

$$4\sin^2\theta = 3 + \cos^2\theta = 3 + 1 - \sin^2\theta \quad \Rightarrow 5\sin^2\theta = 4 \quad \Rightarrow \sin^2\theta = \frac{4}{5} \quad \Rightarrow \sin\theta = \pm\frac{2}{\sqrt{5}}$$

$$\theta = \sin^{-1}\left(\pm\frac{2}{\sqrt{5}}\right) = \sin^{-1}(\pm 0.8944) = 63.43^0, \ 116.565^0, 243.43^0, 296.568^0$$

Ex-19-9: The points (2, 6) and (3, 18) lie on the curve $y = ax^n$.

Use logarithms to find the values of a and n, giving your answers correct to 2 decimal places.

$$y = ax^n$$

$$6 = a(2)^n \qquad \ldots(1)$$

$$18 = a(3)^n \qquad \ldots(2)$$

Dividing Eq.2 by 1:

$$\frac{18}{6} = \frac{a(3)^n}{a(2)^n} \quad \Rightarrow 3 = \frac{(3)^n}{(2)^n} = \left(\frac{3}{2}\right)^n \quad \Rightarrow \log 3 = n\log\left(\frac{3}{2}\right) \Rightarrow n = \frac{\log 3}{\log 1.5} = 2.71$$

$$6 = a(2)^n = a(2)^{2.71} \quad \Rightarrow \log 6 = \log\left(a \times 2^{2.71}\right) = \log a + 2.71\log 2$$

$\Rightarrow \log a = \log 6 - 2.71 \log 2 = 0.778 - 0.816 = -0.0378 \qquad \Rightarrow a = \log^{-1}(-0.0378) = 0.917$

OR

$y = ax^n$

$\log y = \log a + n \log x$

$\log 6 = \log a + n \log 2 \qquad \ldots(1)$

$\log 18 = \log a + n \log 3 \qquad \ldots(2)$

Subtracting Eq.2 from 1

$\log 18 - \log 6 = \log a - \log a + n \log 3 - n \log 2$

$\log 18 - \log 6 = \ln \log 3 - n \log 2$

$\log \dfrac{18}{6} = \log 3^n - \log 2^n = \log\left(\dfrac{3}{2}\right)^n = n\log(1.5) = \log 3$

$\Rightarrow n = \dfrac{\log 3}{\log(1.5)} = 2.71$

$6 = a(2)^n = a(2)^{2.71} \quad \Rightarrow \log 6 = \log\left(a \times 2^{2.71}\right) = \log a + 2.71 \log 2$

$\Rightarrow \log a = \log 6 - 2.71 \log 2 = 0.778 - 0.816 = -0.0378 \qquad \Rightarrow a = \log^{-1}(-0.0378) = 0.917$

SECTION B (P-19)

Ex-19-10: (i) Find the equation of the tangent to the curve $y = x^4$ at the point where $x = 2$. Give your answer in the form $y = mx + c$

$y = x^4$

Gradient= $\dfrac{dy}{dx} = 4x^3 \bigg|_{x\,=\,2} = 32$

$y = mx + c = 32x + c$ now $y = x^4 = 2^4 = 16$

$16 = 32(2) + c \quad \Rightarrow c = 16 - 64 = -48$

$\Rightarrow y = 32x - 48$

(ii) Calculate the gradient of the cord joining the points on the curve $y = x^4$ where $x = 2$ and $x = 2.1$

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y_1 = x^4 \Big|_{x=2} = 16 \qquad P_1(2,16)$$

$$y_2 = x^4 \Big|_{x=2.1} = (2.1)^4 \qquad P_1(2.1, 19.448)$$

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{19.448 - 16}{2.1 - 2.0} = \frac{3.658}{0.1} = 36.58$$

(iii) (A) Expand $(2+h)^4$

$$(2+h)^4 = 2^4 + 4 \times 2^3 h + \frac{4 \times 3 \times 2^2 h^2}{2 \times 1} + \frac{2 \times 4 \times 3 \times 2h^3}{2 \times 3} + h^4 = 2^4 + 32h + 24h^2 + 8h^3 + h^4$$

(B) Simplify $\dfrac{(2+h)^4 - 2^4}{h}$

$$\frac{(2+h)^4 - 2^4}{h} = \frac{2^4 + 32h + 24h^2 + 8h^3 + h^4 - 2^4}{h} = \frac{32h + 24h^2 + 8h^3 + h^4}{h} = 32 + 24h + 8h^2 + h^3$$

(C) Show how your result in part (iii) (B) can be used to find the gradient of $y = x^4$ at the point where $x = 2$

Ex-19-11: (a) A boat travels from P to Q and then to R. As shown in Fig-19-11-1, Q ia 10.6 km from P on a bearing of 45^0. R is 9.2 km from P on a bearing of 113^0, so that angle QPR is 68^0. Calculate the distance and bearing of R from Q.
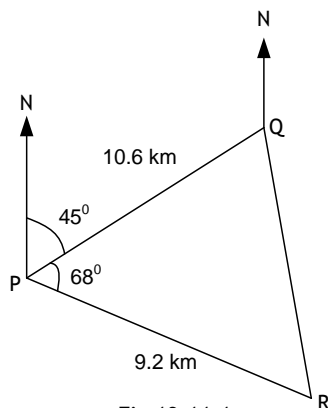
Fig-19-11-1

$$QR^2 = (10.6)^2 + (9.2)^2 - 2(10.6)(9.2)\cos 68^0 = 112.36 + 84.64 - 73.06 = 123.94$$

$$\Rightarrow QR = \sqrt{123.94} = 11.133\,km$$
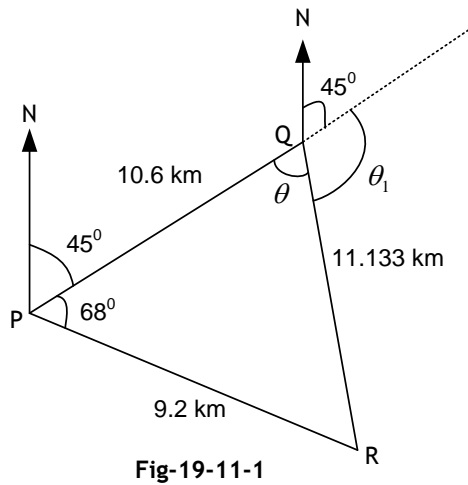
Fig-19-11-1

$$\frac{11.133}{\sin 68^0} = \frac{9.2}{\sin \theta} \Rightarrow \sin \theta = \frac{9.2 \times \sin 68^0}{11.133} = 0.766$$

$$\Rightarrow \theta = \sin^{-1}(0.766) = 50^0$$

$$\Rightarrow \theta_1 = 180^0 - 50^0 = 130^0$$

$$Bearing\ of\ R\ from\ Q = 45 + 130 = 175^0$$

(b) Fig-19-11-2 shows the cross-section, EBC, of the rudder of a boat.
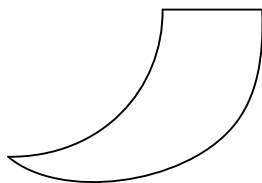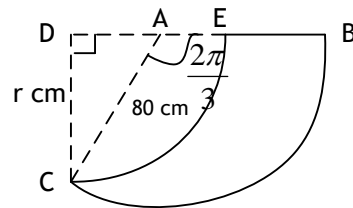
rudder   **Fig-19-11-2**   Detail of construction

BC is an arc of a circle with centre A and radius 80 cm. Angle $CAB = \dfrac{2\pi}{3}$ radians.

EC is an arc of a circle with centre D and radius r cm. Angle CDE is a right angle.
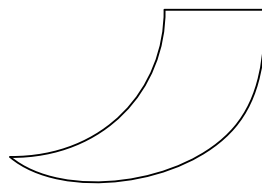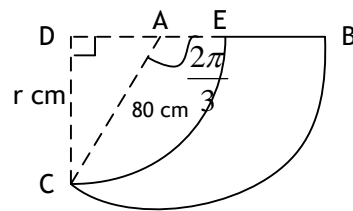
rudder   **Fig-19-11-2**   Detail of construction

(i) Calculate the area of a sector ABC.

$$Area\ of\ ABC = \frac{1}{2}r^2\theta = \frac{1}{2}(80)^2\left(\frac{2\pi}{3}\right) = 6702.06\ cm^2$$

(ii) Show that $r = 40\sqrt{3}$ and calculate the area of triangle CDA.
Ans:

149
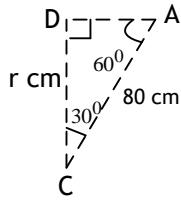
$$\sin 60^0 = \frac{r}{80} \Rightarrow r = 80 \sin 60^0 = 80 \times \frac{\sqrt{3}}{2} = 40\sqrt{3} \ cm$$

(iii)   Hence calculate the area of cross-section of the rudder.

Area of the triangle$= \frac{1}{2} CD \times AC \times \sin 30^0 = \frac{1}{2} \times 40\sqrt{3} \times 80 = 2771.3 \ cm^2$

$Area \ of \ DEC = \frac{1}{2} r^2 \theta = \frac{1}{2}(40\sqrt{3})^2 \times \frac{\pi}{2} = 3769.9 \ cm^2$

$Area \ of \ AEC = Area \ of \ DEC - DAC = 3769.9 - 2771.3 = 998.6 \ cm^2$

$Area \ of \ EBC = Area \ of \ ABC - AEC = 6702.1 - 998.6 = 5703.5 \ cm^2$

PAPER-20

SECTION A (C2-6-6-06

Ex-20-1:   Write down the value of $\log_a a$ and $\log_a (a)^3$

Ans: $\log_a a = 1$, and $\log_a (a)^3 = 3\log_a a = 3$

Ex-20-2:   The first term of a geometric series is 8. The sum to infinity of the series is 10. Find the common ratio.

Ans: $S_\infty = \frac{a_1}{1-r} = \frac{8}{1-r} = 10 \quad \Rightarrow 10 - 10r = 8 \quad \Rightarrow -10r = 8 - 10 = -2 \quad \Rightarrow r = \frac{1}{5}$

Ex-20-3: θ is an acute angle and $\sin \theta = \frac{1}{4}$. Find the exact value of $\tan \theta$.

Ans:

$\tan \theta = \frac{1}{\sqrt{15}}$

$\sqrt{16-1} = \sqrt{15}$
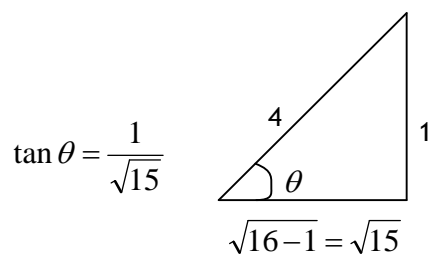
Ex-20-4: Find $\int_{1}^{2}\left(x^4 - \dfrac{3}{x^2} + 1\right)dx$, showing your working.

Ans:

$$\int_{1}^{2}\left(x^4 - \frac{3}{x^2} + 1\right)dx = \int_{1}^{2}(x^4 - 3x^{-2} + 1)dx = \frac{x^5}{5} - 3\frac{x^{-2+1}}{-1} + x = \frac{x^5}{5} + \frac{3}{x} + x\Big|_{1}^{2} = \frac{2^5}{5} + \frac{3}{2} + 2 - \left(\frac{1^5}{5} + \frac{3}{1} + 1\right)$$

$$= 9.9 - 4.2 = 5.7$$

Ex-20-5: The gradient of a curve is given by $\dfrac{dy}{dx} = 3 - x^2$. The curve passes through the point (6, 1). Find the equation of the curve.

Ans:

$$\frac{dy}{dx} = 3 - x^2 \Rightarrow dy = (3 - x^2)dx \Rightarrow \int dy = \int(3 - x^2)dx \Rightarrow y = 3x - \frac{x^3}{3} + c \Rightarrow 1 = 3(6) - \frac{6^3}{3} + c \Rightarrow c = 55$$

$$\Rightarrow y = 3x - \frac{x^3}{3} + 55$$

Ex-20-6: A sequence is given by the following.

$$u_1 = 3.$$

$$u_{n+1} = u_n + 5$$

(i) Write down the first 4 terms of this sequence.

Ans: $a_1 = 3$

$$a_{1+1} = a_2 = a_1 + 5 = 3 + 5 = 8$$

$$a_{2+1} = a_3 = a_2 + 5 = 8 + 5 = 13$$

$$a_{3+1} = a_4 = a_3 + 5 = 13 + 5 = 18$$

(ii) Find the sum of the 51st to the 100th terms, inclusive, of the sequence.

Ans: It is an arithmetic series as the difference, d=13-8=5

$$S_n = \frac{n}{2}(2a + (n-1)d) \Rightarrow S_{50} = \frac{50}{2}[2 \times 3 + 5(50 - 1)] = 25[6 + 245] = 6275$$

Ex-20-7: (i) Sketch the graph of $y = \cos x$ for $0^0 \le x \le 360^0$.

Ans:
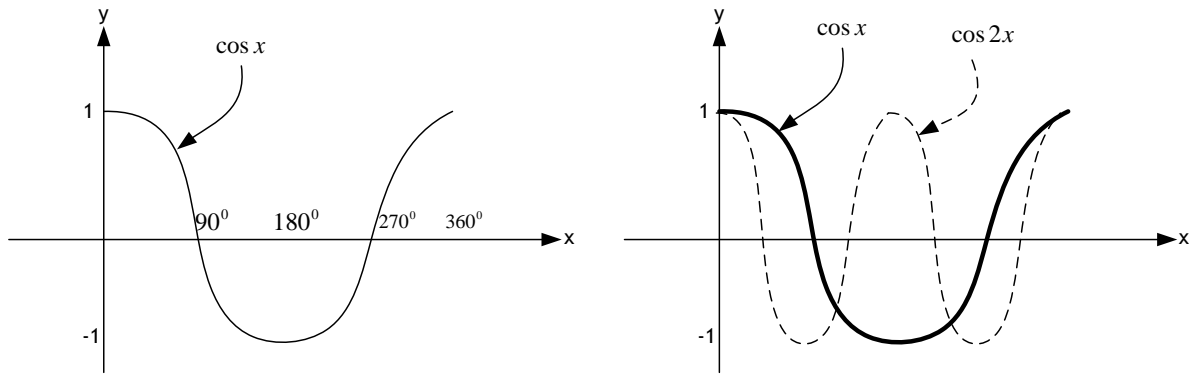
On the same axes, sketch the graph of $y = \cos 2x$ for $0^0 \leq x \leq 360^0$. Label each graph clearly.

(ii)   Solve the equation $\cos 2x = 0.5$ for $0^0 \leq x \leq 360^0$.

Ans: $\cos 2x = 0.5 \quad \Rightarrow 2x = 60^0,\ 300^0,\ 420^0,\ 660^0 \quad \Rightarrow x = 30^0,\ 150^0,\ 210^0,\ 330^0$

Ex-20-8:   Given that for $y = 6x^3 + \sqrt{x} + 3$, find $\dfrac{dy}{dx}$ *and* $\dfrac{d^2 y}{dx^2}$

Ans: $y = 6x^3 + \sqrt{x} + 3 \quad \Rightarrow \dfrac{dy}{dx} = 18x^2 + \dfrac{1}{2}x^{-\frac{1}{2}} = 18x^2 + \dfrac{1}{2\sqrt{x}}$

$\Rightarrow \dfrac{dy}{dx} = 18x^2 + \dfrac{1}{2\sqrt{x}} \qquad \Rightarrow \dfrac{d^2 y}{dx^2} = 36x - \dfrac{1}{4}x^{-\frac{3}{2}}$

Ex-20-9:   Use logarithms to solve the equation $5^{3x} = 100$. Give your answer correct to 3 decimal places.

Ans: $\log_{10} 5^{3x} = \log_{10} 100 = 2 \qquad \Rightarrow 3x \log_{10} 5 = 2 \qquad \Rightarrow x = \dfrac{2}{3\log_{10} 5} = 0.954$

SECTION B (P-20)

Ex-20-10: (i)   At a certain time, ship S is 5.2 km from lighthouse L on a bearing of 48^0. At the same time, ship T is 6.3 km from L on a bearing of 105^0, as shown in Fig-20-10-1. For these positions, calculate

Fig-20-10-1
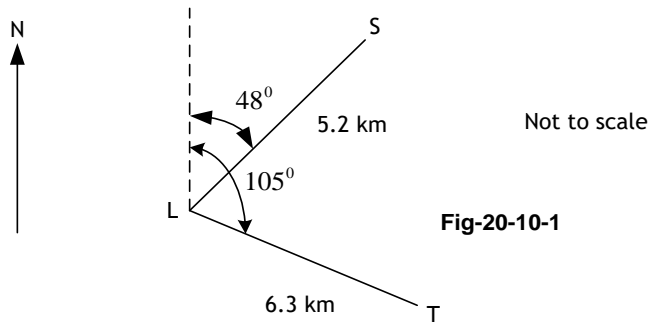
(A) the distance between ships S and T,

Not to scale
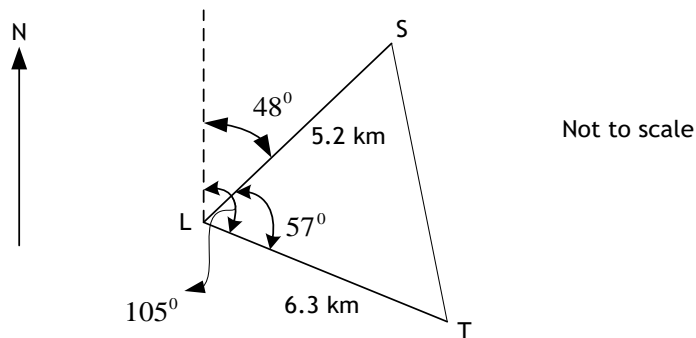
$$ST^2 = (5.2)^2 + (6.3)^2 - 2(5.2)(6.3)\cos 57^0 = 27.04 + 39.69 + 65.52\cos 57^0 = 31.04$$
$$ST^2 = 31.04$$

$$ST = 5.57$$

(B) the bearing of S from T.

Not to scale
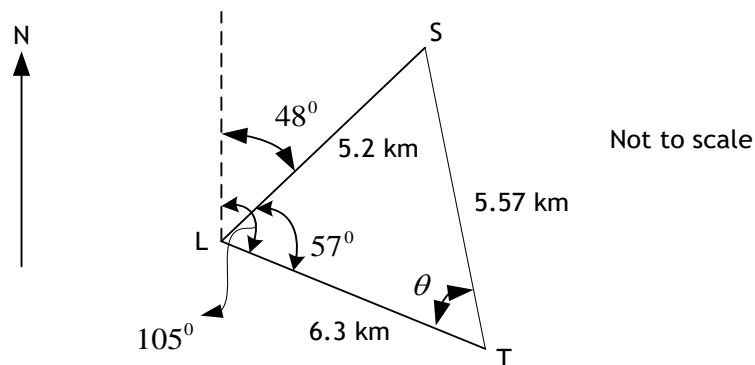
$$\frac{5.57}{\sin(57^0)} = \frac{5.2}{\sin\theta} \quad \Rightarrow \sin\theta = \frac{5.2\sin 57^0}{5.57} = 0.783 \quad \Rightarrow \theta = 51.53^0, \text{ a}$$
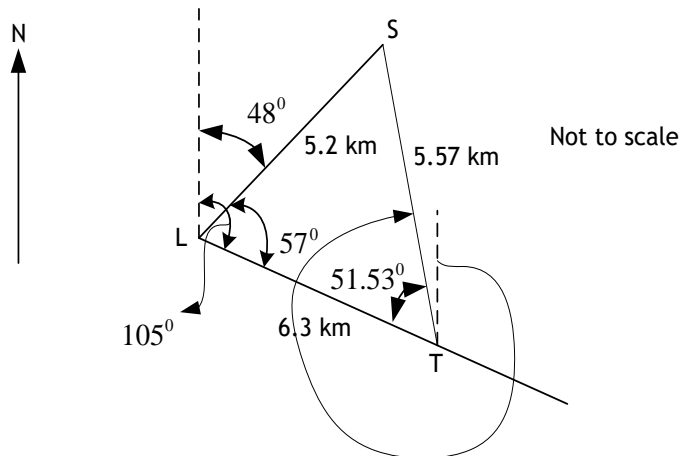
$$Bearing\ of\ S\ from\ T = 105^0 + 180^0 + 51.53^0 = 336.53^0$$

(ii)

Fig-20-10-2
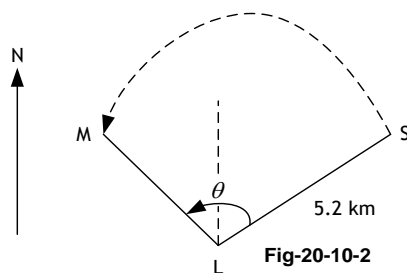
Ship S then travels at 24 kmh^{-1} anticlockwise along the arc of a circle,

keeping 5.2 km from the lighthouse L, as shown in Fig.10.2.

Find, in radians, the angle θ that the line LS has turned through in 26 minutes.

Ans:

$$SM = 24\frac{km}{hr} \times \frac{26}{60} hr = 10.4\,km \quad \Rightarrow SM = r\theta = 5.2\theta = 10.4 \quad \Rightarrow \theta = \frac{10.4}{5.2} = 2\,rad = \frac{2(180)}{\pi} = 114.59^0$$

$$\theta_4 = 114.59^0 - 48^0 = 67^0$$
$$\theta_2 = 90^0 - \theta_1 = 90^0 - 48^0 = 42^0$$
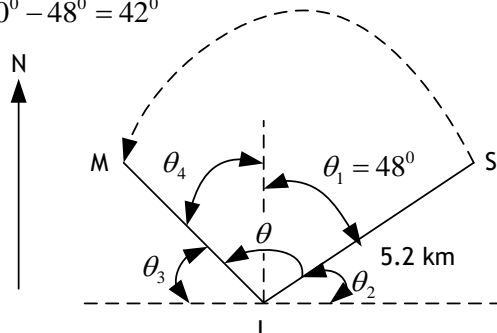
$$\theta_3 = 180^0 - 67^0 - 48^0 - 42^0 = 23^0$$ **Fig.-20-10-2**

Hence find, in degrees, the bearing of ship S from the lighthouse at this time.

Bearing of ship S from the lighthouse $= 180^0 + 48^0 + 42^0 + 23^0 = 293^0$

Ex-20-11: A cubic curve has equation $y = x^3 - 3x^2 + 1$.

    (i)   Use calculus to find the coordinates of the turning points on this curve. Determine the nature of these turning points.

$y = x^3 - 3x^2 + 1$

$\dfrac{dy}{dx} = 3x^2 - 6x = 0 \quad \Rightarrow 3x(x-2) = 0 \quad \Rightarrow x = 0 \quad and \quad x = 2$

$\dfrac{d^2y}{dx^2} = 6x - 6\Big|_{x=0} = -6 \Rightarrow Max\ (0,1) \quad and \quad \dfrac{d^2y}{dx^2} = 6x - 6\Big|_{x=2} = 12 - 6 = 6 \Rightarrow Min\ (2,-3)$

    (ii)   Show that the tangent to the curve at the point where x = -1 has gradient 9.

$gradient = \dfrac{dy}{dx}\Big|_{x=-1} = 3x^2 - 6x\Big|_{x=-1} = 3(-1)^2 - 6(-1) = 3 + 6 = 9$

    Find the coordinates of the other point, P, on the curve at which the tangent has gradient 9 and find the equation of the normal to the curve at P.

$gradient = \dfrac{dy}{dx} = 3x^2 - 6x = 9 \quad \Rightarrow x^2 - 2x - 3 = 0 \quad \Rightarrow (x-3)(x+1) = 0 \quad \Rightarrow x = 3 \ other\ point$

$y = x^3 - 3x^2 + 1\Big|_{x=3} = 27 - 27 + 1 = 1$

$P(3,1) \qquad \Rightarrow m = -\dfrac{1}{9} \ Perpendicular$

$y = mx + c = -\dfrac{1}{9}x + c \qquad \Rightarrow 1 = -\dfrac{1}{9}(3) + c \Rightarrow c = \dfrac{4}{3}$

$y = -\dfrac{1}{9}x + \dfrac{4}{3}$

Show that the area of the triangle bounded by the normal at P and the x- and y-axes is 8 square units.
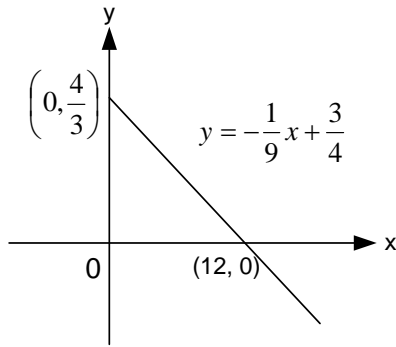
$$Area = \frac{1}{2}(12)\left(\frac{4}{3}\right) = 8$$

Ex-20-12:   Answer the whole of this question on the insert provided.

A colony of bats is increasing. The population, P, is modeled by $P = a \times 10^{bt}$, where t is the time in years after 2000.

(i)   Show that, according to this model, the graph of $\log_{10} P$ against t should be a straight line of gradient b. State, in terms of a, the intercept on the vertical axis.

$$P = a \times 10^{bt}$$

$$\log P = \log(a \times 10^{bt}) = \log a + bt \log 10 = bt + \log a$$

$$\log P = bt + \log a$$

$$y = mx + c$$

(ii)   The table gives the data for the population from 2001 to 2005.

Year	2001	2002	2003	2004	2005
t	1	2	3	4	5
P	7900	8800	10000	11300	12800

Complete the table of values on the insert, and plot $\log_{10} P$ against t. Draw a line of best fit for the data.

(iii)   Use your graph to find the equation for P in terms of t.

(iv)   Predict the population in 2008 according to this model.

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Year	2001	2002	2003	2004	2005
t	1	2	3	4	5
P	7900	8800	10000	11300	12800
logP	3.897	3.944	4	4.053	4.107
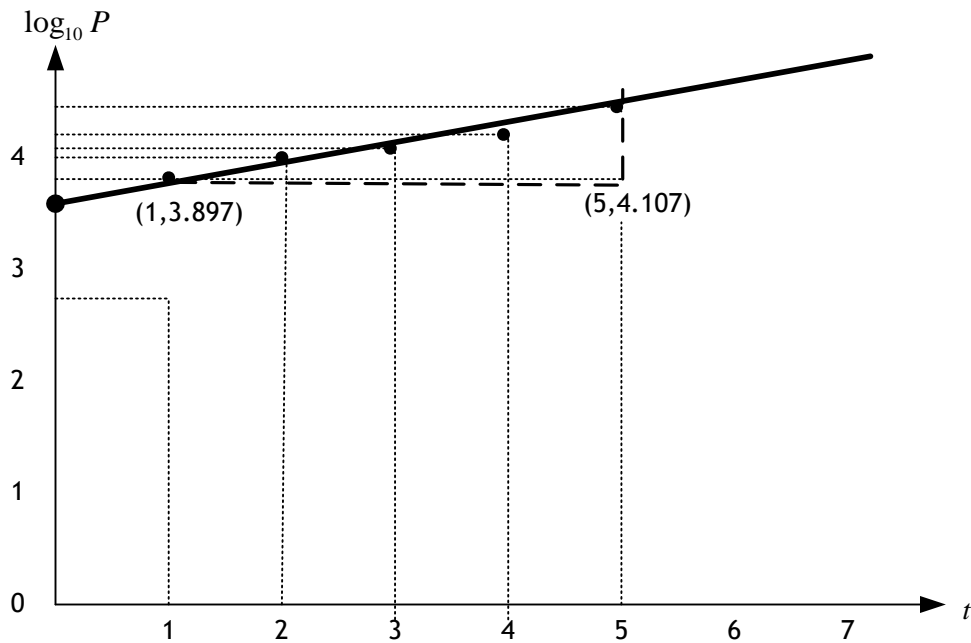
$\log P = bt + \log a$

$\log a = c = \cong 3.70 \quad \Rightarrow a = 5012$

$b = gradient = m = \dfrac{4.107 - 3.897}{5 - 1} = \dfrac{0.273}{4} = 0.0682$

$P = a \times 10^{bt}$

$P = a \times 10^{bt} = 5012 \times 10^{0.0682t}$

$P = 5012 \times 10^{0.0682t}$

$P = a \times 10^{bt} = 5012 \times 10^{0.0682t}$

$P = 5012 \times 10^{0.0682(8)} = 17604$

Year	2001	2002	2003	2004	2005
t	1	2	3	4	5
P	7900	8800	10000	11300	12800
$\log_{10} P$	3.897	3.944	4	4.053	4.107
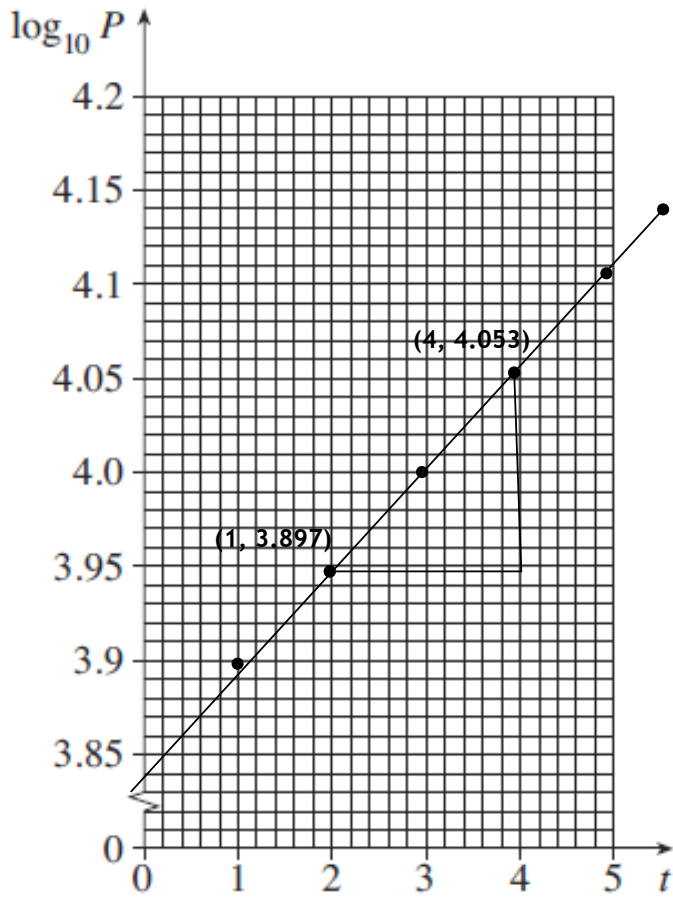
$\log P = bt + \log a$

$\log a = c \cong 3.84 \implies a = 6918$

$$b = gradient = m = \frac{4.053 - 3.897}{4 - 1} = \frac{0.273}{3} = 0.052$$

$P = a \times 10^{bt}$

$P = a \times 10^{bt} = 6918 \times 10^{0.052t}$

$P = 6918 \times 10^{0.052t}$

$P = a \times 10^{bt} = 6918 \times 10^{0.052t}$

$P = 6918 \times 10^{0.052(8)} = 18029.4$

PAPER-21

DERIVATIVES

Ex-21-1: $y = au^n$

Ans: $\dfrac{dy}{dx} = nau^{n-1}\dfrac{du}{dx}$

Ex-21-2: Find the derivative of $y = 3x^2$.

Ans: $\dfrac{dy}{dx} = 6x$.

Ex-21-3: Find the derivative of $y = 4x^4 + 6x^3 + 3x^2 + 8x + 34$.

Ans: $\dfrac{dy}{dx} = 16x^3 + 18x^2 + 6x + 8$.

Ex-21-4: Find the derivative of $y = 3x^2 + 7x + 8$.

Ans: $\dfrac{dy}{dx} = 6x + 7$.

Ex-21-5: Find the derivative of $y = (x+4)^3$.

Ans: $\dfrac{dy}{dx} = 3(x+4)^2$.

Ex-21-6: Find the derivative of $y = (3x+4)^3$.

Ans: $\dfrac{dy}{dx} = 3(3x+4)^2 \times 3 = 9(3x+4)^2$.

OR

Let $u = 3x + 4 \qquad \Rightarrow y = u^3$

$\dfrac{dy}{dx} = 3u^2 \times \dfrac{du}{dx} = 3(3x+4)^2 \times 3 = 9(3x+4)^2$

Ex-21-7: Find the derivative of $y = \sqrt{x}$

Ans: $y = \sqrt{x} = x^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{x}}$

Ex-21-8: Find the derivative of $y = \sqrt{x^2 + 1}$

Ans: $y = \sqrt{x^2 + 1} = (x^2 + 1)^{\frac{1}{2}} \qquad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2}(x^2+1)^{-\frac{1}{2}} \times 2x = \dfrac{x}{\sqrt{x^2+1}}$

Ex-21-9: Find the derivative of $y = \dfrac{1}{\left(x^2 + 3x + 2\right)^3}$

Ans: $y = \dfrac{1}{\left(x^2 + 3x + 2\right)^3} = \left(x^2 + 3x + 2\right)^{-3}$

$\dfrac{dy}{dx} = -3\left(x^2 + 3x + 2\right)^{-4} \times (2x + 3) = -\dfrac{3(2x + 3)}{\left(x^2 + 3x + 2\right)^4}$

Ex-21-10: Find the derivative of $y = \left(x^2 + 3x + 2\right)^{-3}$

Ans: $y = \left(x^2 + 3x + 2\right)^{-3}$

$\dfrac{dy}{dx} = -3\left(x^2 + 3x + 2\right)^{-4} \times (2x + 3) = -\dfrac{3(2x + 3)}{\left(x^2 + 3x + 2\right)^4}$

Ex-21-11: Find the derivative of $y = \dfrac{x^2 + 3x}{\sqrt{x}}$

Ans: $y = \dfrac{x^2 + 3x}{\sqrt{x}} = \dfrac{x^2}{\sqrt{x}} + \dfrac{3x}{\sqrt{x}} = \dfrac{x^2}{x^{\frac{1}{2}}} + \dfrac{3x}{x^{\frac{1}{2}}} = x^{2-\frac{1}{2}} + 3x^{1-\frac{1}{2}} = x^{\frac{3}{2}} + 3x^{\frac{1}{2}}$

$y = x^{\frac{3}{2}} + 3x^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \dfrac{3}{2}x^{\frac{3}{2}-1} + 3 \times \dfrac{1}{2}x^{\frac{1}{2}-1}$

Ex-21-12: Find the derivative of $y = \ln(3x + 4)$

Note: $y = \ln u$

$\dfrac{dy}{dx} = \dfrac{1}{u} \times \dfrac{du}{dx}$

Ans: $\dfrac{dy}{dx} = \dfrac{1}{3x + 4} \times 3 = \dfrac{3}{3x + 4}$

Ex-21-13: Find the derivative of $y = \ln\left(4 - 5x^3\right)$

Note: $y = \ln u$

$\dfrac{dy}{dx} = \dfrac{1}{u} \times \dfrac{du}{dx}$

Ans: $\dfrac{dy}{dx} = \dfrac{1}{4 - 5x^3} \times \left(-15x^2\right) = -\dfrac{15x^2}{4 - 5x^3}$

Ex-21-14: Find the derivative of $y = \log\left(4 - 5x^3\right)$

Note: $y = \log u$

$\dfrac{dy}{dx} = \dfrac{1}{u} \times \dfrac{du}{dx} \times \dfrac{1}{\ln 10}$

Ans: $\dfrac{dy}{dx} = \dfrac{1}{4-5x^3} \times \left(-15x^2\right) \times \dfrac{1}{\ln 10} = -\dfrac{15x^2}{\ln 10 \times \left(4-5x^3\right)}$

Ex-21-15: Find the derivative of $y = \sin\left(2x + 45^0\right)$

Note: $y = \sin u$

$\dfrac{dy}{dx} = \cos u \times \dfrac{du}{dx}$

Ans: $\dfrac{dy}{dx} = \cos\left(2x + 45^0\right) \times 2 = 2\cos\left(2x + 45^0\right)$

Ex-21-16: Find the derivative of $y = \cos\left(5x + 25^0\right)$

Note: $y = \cos u$

$\dfrac{dy}{dx} = -\sin u \times \dfrac{du}{dx}$

Ans: $\dfrac{dy}{dx} = -\sin\left(5x + 25^0\right) \times 5 = -5\sin\left(5x + 25^0\right)$

Ex-21-17: Find the derivative of $y = \dfrac{\sin 2x}{\cos 2x}$

Note: $y = \dfrac{u}{v}$

$\dfrac{dy}{dx} = \dfrac{v \times \dfrac{du}{dx} - u \times \dfrac{dv}{dx}}{v^2}$

Ans: $\dfrac{dy}{dx} = \dfrac{\cos 2x \times 2 \times \cos 2x - \sin 2x\left(-\sin 2x\right) \times 2}{\left(\cos 2x\right)^2}$

$\dfrac{dy}{dx} = \dfrac{2\left(\cos 2x\right)^2 + 2\left(\sin 2x\right)^2}{\left(\cos 2x\right)^2} = \dfrac{2\left[\left(\sin 2x\right)^2 + \left(\cos 2x\right)^2\right]}{\left(\cos 2x\right)^2} = \dfrac{2}{\left(\cos 2x\right)^2} = 2\left(\sec 2x\right)^2$

Ex-21-18: Find the derivative of $y = x \ln x$

Note: $y = u \times v$

$\dfrac{dy}{dx} = u \times \dfrac{dv}{dx} + v \times \dfrac{du}{dx}$

Ans: $\dfrac{dy}{dx} = x \times \dfrac{1}{x} + \ln x = 1 + \ln x$

Ex-21-19: Find the derivative of $y = e^{4x}$

Note:

$$y = e^u$$

$$\frac{dy}{dx} = e^u \times \frac{du}{dx} = e^{4x} \times 4 = 4e^{4x}$$

PAPER-22

SECTION A (C3-25-5-05)

Ex-22-1: Solve the equation $|3x+2| = 1$

$$|3x+2| = 1$$

$$(3x+2) = \pm 1$$

a. $(3x+2) = 1 \Rightarrow 3x = 1-2 = -1 \Rightarrow x = -\frac{1}{3}$

b. $(3x+2) = -1 \Rightarrow 3x+2 = -1 \Rightarrow 3x = -3 \Rightarrow x = -1$

Ex-22-2: Given that $\arcsin x = \frac{\pi}{6}$, find x. Find $\arccos x$ in terms of π.

$$\arcsin x = \frac{\pi}{6} \Rightarrow x = \sin\left(\frac{\pi}{6}\right) = 0.5$$

$$\arccos x = \arccos(0.5) = \frac{\pi}{3}$$

Ex-22-3: The functions $f(x)$ and $g(x)$ are defined for the domain $x > 0$ as follows:

$$f(x) = \ln x, \qquad g(x) = x^3$$

Express the composite function $fg(x)$ in terms of $\ln x$

$$fg(x) = f(x^3) = \ln x^3 = 3\ln x$$

State the transformation which maps the curve $y = f(x)$ onto the curve $y = fg(x)$.

Ex-22-4: The temperature $T^0 C$ of a liquid at time t minutes is given by the equation $T = 30 + 20e^{-0.05t} \quad for\ t \geq 0$.

Write down the initial temperature of the liquid, and find the initial rate of change of temperature.

Find the time at which the temperature is $40^0 C$.

$$T = 30 + 20e^{-0.05t}\Big|_{t=0} = 30 + 20 = 50^0 C$$

$$\frac{dT}{dt} = 20\left(-0.05e^{-0.05t}\right)\Big|_{t=0} = -1\frac{{}^0 C}{s}$$

$$40 = 30 + 20e^{-0.05t} \Rightarrow 20e^{-0.05t} = 40 - 30 = 10 \Rightarrow 20e^{-0.05t} = 10 \Rightarrow e^{-0.05t} = \frac{10}{20} = 0.5$$

$$\Rightarrow -0.05t = \ln 0.5 = -0.693 \Rightarrow t = \frac{-0.693}{-0.05} = 13.863 \ \min utes$$

Ex-22-5: Using the substitution $u = 2x+1$, show that $\int_0^1 \frac{x}{2x+1}dx = \frac{1}{4}(2-\ln 3)$

$$\int_0^1 \frac{x}{2x+1}dx$$

$$u = 2x+1 \quad \Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow dx = \frac{du}{2}$$

$$u = 2x+1 \quad \Rightarrow 2x = u-1 \quad \Rightarrow x = \frac{u-1}{2}$$

$$\int_0^1 \frac{x}{2x+1}dx = \int \left(\frac{u-1}{2}\right)\left(\frac{1}{u}\right)\frac{du}{2} = \frac{1}{4}\int \frac{u-1}{u}du = \frac{1}{4}\int \frac{udu}{u} - \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}\int du - \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}u - \frac{1}{4}\ln u$$

$$\int_0^1 \frac{x}{2x+1}dx = \frac{1}{4}(2x+1)\Big|_0^1 - \frac{1}{4}\ln(2x+1)\Big|_0^1 = \frac{1}{4}(3-1) - \frac{1}{4}(\ln 3 - \ln 1) = \frac{1}{4}(2-\ln 3)$$

Ex-22-6: A curve has equation $y = \dfrac{x}{2+3\ln x}$. Find $\dfrac{dy}{dx}$. Hence find the exact coordinates of the stationary point of the curve.

$$y = \frac{x}{2+3\ln x} \quad \Rightarrow \frac{dy}{dx} = \frac{(2+3\ln x) - x\left(\dfrac{3}{x}\right)}{(2+3\ln x)^2} = \frac{(2+3\ln x)-3}{(2+3\ln x)^2} = \frac{3\ln x - 1}{(2+3\ln x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{3\ln x - 1}{(2+3\ln x)^2} = 0 \quad \Rightarrow 3\ln x - 1 = 0 \quad \Rightarrow 3\ln x = 1 \quad \Rightarrow \ln x = \frac{1}{3} \quad \Rightarrow x = e^{\frac{1}{3}}$$

$$y = \frac{x}{2+3\ln x}\Big|_{x=e^{\frac{1}{3}}} = \frac{e^{\frac{1}{3}}}{2+3\ln e^{\frac{1}{3}}} = \frac{e^{\frac{1}{3}}}{2+\dfrac{3}{3}\ln e} = \frac{e^{\frac{1}{3}}}{2+1} = \frac{1}{3}e^{\frac{1}{3}} \quad \Rightarrow Po\,int\left(e^{\frac{1}{3}}, \frac{1}{3}e^{\frac{1}{3}}\right)$$

Ex-22-7: Fig-22-7 shows the curve defined implicitly by the equation $y^2 + y = x^3 + 2x$, together with line $x = 2$.

Fig-22-7
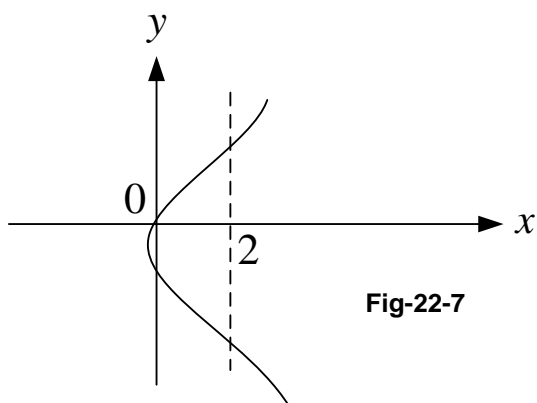
Find the coordinates of the points of intersection of the line and the curve.

Find $\dfrac{dy}{dx}$ in terms of x and y. Hence find the gradient of the curve at each of these two points.

$$y^2 + y = x^3 + 2x\Big|_{x=2} = 2^3 + 2(2) = 12$$

$$\Rightarrow y^2 + y = 12 \quad \Rightarrow y^2 + y - 12 = 0 \quad \Rightarrow (y+4)(y-3) = 0 \quad \Rightarrow y = -4 \quad and \quad y = 3$$
$$\Rightarrow P_1(2,3) \quad and \quad P_2(2,-4)$$

$$y^2 + y = x^3 + 2x \quad \Rightarrow 2y\frac{dy}{dx} + \frac{dy}{dx} = 3x^2 + 2 \quad \Rightarrow (2y+1)\frac{dy}{dx} = 3x^2 + 2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x^2 + 2}{2y+1}$$

$$\Rightarrow \frac{dy}{dx}\Big|_{at\ p_1} = gradient = \frac{3x^2 + 2}{2y+1}\Big|_{x=2,\ y=3} = \frac{12+2}{6+1} = \frac{14}{7} = 2$$

$$\Rightarrow \frac{dy}{dx}\Big|_{at\ p_2} = gradient = \frac{3x^2 + 2}{2y+1}\Big|_{x=2,\ y=-4} = \frac{12+2}{-8+1} = \frac{14}{-7} = -2$$

SECTION B(P-22)

Ex-22-8: Fig-22-8 shows part of the curve $y = x\sin 3x$. It crosses the x-axis at P. The point on the curve with x-coordinate $\dfrac{1}{6}\pi$ is Q.
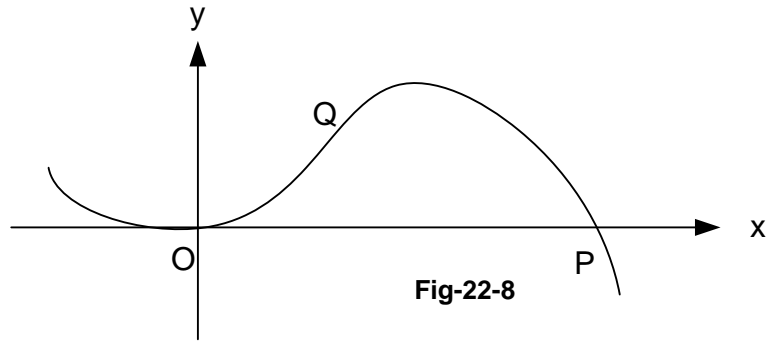
Fig-22-8

(i) Find the x-coordinate of P.

$$y\Big|_{at\ P} = 0 \implies y = x\sin 3x = 0 \implies x = 0 \quad and \quad \sin 3x = 0 \implies 3x = 0,\ \pi$$

$$\implies x = \frac{\pi}{3}$$

(ii) Show that Q lies on the $y = x$

$$y = x\sin 3x\Big|_{x=\frac{\pi}{6}} = \frac{\pi}{6}\sin\left(\frac{3\pi}{6}\right) = \frac{\pi}{6}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{6}$$

Yes, Q lies on the $y = x$, because both x and y are equal to $\frac{\pi}{6}$

(iii) Differentiate $x\sin 3x$. Hence prove that the line $y = x$ touches the curve at Q

$$\frac{d}{dx}(x\sin 3x) = 3x\cos 3x + \sin 3x\Big|_{x=\frac{\pi}{6}} = \frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) = 1$$

$$y = x \implies \frac{dy}{dx} = 1$$

(iv) Show that the area of the region bounded by the curve and the line $y = x$ is $\frac{1}{72}(\pi^2 - 8)$

y=x

Fig-22-8

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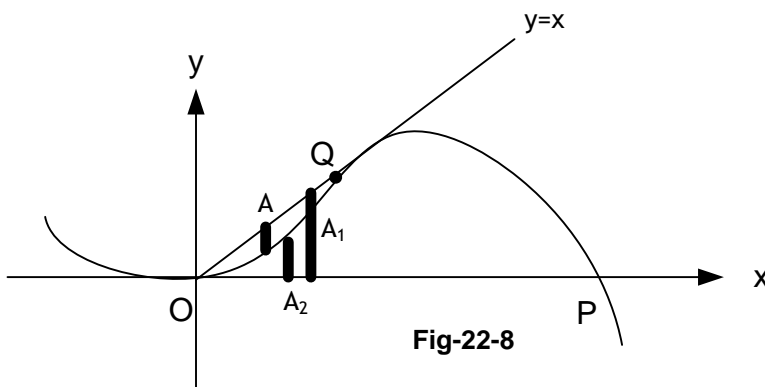
$$A_1 = \frac{1}{2}\left(\frac{\pi}{6}\right)\left(\frac{\pi}{6}\right) = \frac{\pi^2}{72}$$

$$A_2 = \int_0^{\frac{\pi}{6}} x \sin 3x\, dx = \frac{1}{9}$$

$$A = A_1 - A_2 = \frac{\pi^2}{72} - \frac{1}{9} = \frac{\pi^2 - 8}{72}$$

Ex-22-9:   The function $f(x) = \ln\left(1 + x^2\right)$ has domain $-3 \le x \le 3$
Fig-22-9 shows the graph of $y = f(x)$
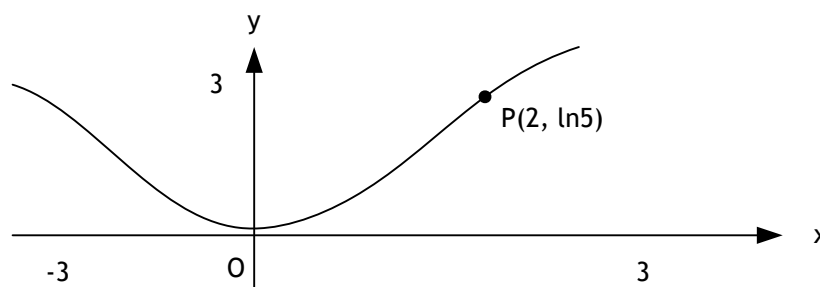
Fig-22-9

(i)   Show algebraically that the function is even. State how this property relates to the shape of the curve.

$$f(x) = \ln\left(1 + x^2\right)$$
$$f(-x) = \ln\left(1 + x^2\right)$$
$$\Rightarrow f(-x) = f(x) \quad \Rightarrow even$$

(ii)   Find the gradient of the curve at the point P(2, ln5).

$$y = f(x) = \ln\left(1 + x^2\right)$$
$$\frac{dy}{dx} = \frac{2x}{1 + x^2}\bigg|_{x=2} = \frac{4}{1+4} = \frac{4}{5} = m$$

(iii)   Explain why the function does not have an inverse for the domain $-3 \le x \le 3$

The domain of $f(x)$ is now restricted to $0 \le x \le 3$. The inverse of $f(x)$ is the function $g(x)$

(iv)   Sketch the curves $y = f(x)$ and $y = g(x)$ on the same axes
State the domain of the function $g(x)$

Show that of $g(x) = \sqrt{e^x - 1}$

$y = f(x) = \ln\left(1 + x^2\right)$

$1 + x^2 = e^y \quad \Rightarrow x^2 = e^y - 1 \quad \Rightarrow x = \sqrt{e^y - 1}$

$\Rightarrow f^{-1}(y) = \sqrt{e^y - 1}$

$\Rightarrow f^{-1}(x) = g(x) = \sqrt{e^x - 1}$

(v)   Differentiate $g(x)$. Hence verify that $g'(\ln 5) = 1\frac{1}{4}$. Explain the connection between this result and your answer to part (ii)

$g(x) = \sqrt{e^x - 1} = \left(e^x - 1\right)^{\frac{1}{2}} \quad \Rightarrow \dfrac{dg}{dx} = \dfrac{1}{2}\left(e^x - 1\right)^{-\frac{1}{2}}\left(e^x\right) = \dfrac{1}{2}\dfrac{e^x}{\sqrt{e^x - 1}}$

$g'(x) = \dfrac{dg}{dx} = \dfrac{1}{2}\dfrac{e^x}{\sqrt{e^x - 1}}\bigg|_{x = \ln 5} = \dfrac{1}{2}\dfrac{e^{\ln 5}}{\sqrt{e^{\ln 5} - 1}} = \dfrac{1}{2}\dfrac{5}{\sqrt{5 - 1}} = \dfrac{1}{2}\dfrac{5}{\sqrt{4}} = \dfrac{5}{4} = 1\dfrac{1}{4}$

The answer in part (ii) is the same as this.

PAPER-23

SECTION A (C3-18-1-06)

Ex-23-1:   Given that $y = (1 + 6x)^{\frac{1}{3}}$, show that $\dfrac{dy}{dx} = \dfrac{2}{y^2}$.

$$y = (1 + 6x)^{\frac{1}{3}} \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3}(1 + 6x)^{-\frac{2}{3}}(6) = 2(1 + 6x)^{-\frac{2}{3}} = \dfrac{2}{(1 + 6x)^{\frac{2}{3}}} = \dfrac{2}{y^2}$$

Ex-23-2:   A population is P million at time t years. P is modelled by the equation

$P = 5 + ae^{-bt}$

Where a and b are constants.

The population is initially 8 million, and declines to 6 million after 1 year.

(i)   Use this information to calculate the values of a and b, giving b correct to 3 significant figures.

$P = 5 + ae^{-bt}\bigg|_{t = 0} = 5 + a = 8 \quad \Rightarrow a = 8 - 5 = 3$

$$P = 5 + 3e^{-bt}\Big|_{t=1} = 5 + 3e^{-b} = 6 \quad \Rightarrow 3e^{-b} = 6 - 5 = 1$$

$$\Rightarrow 3e^{-b} = 1 \quad \Rightarrow e^{-b} = \frac{1}{3} \quad \Rightarrow -b \ln e = \ln\left(\frac{1}{3}\right) = \ln 1 - \ln 3 = 0 - \ln 3$$

$$\Rightarrow -b = -\ln 3 \quad \Rightarrow b = \ln 3$$

(ii)   What is the long-term population predicted by the model?

$$P = 5 + 3e^{-\ln 3t}\Big|_{t \to \infty} = 5 + 0 = 5$$

Ex-23-3:   (i)   Express $2\ln x + \ln 3$ as a single logarithm.

$$2\ln x + \ln 3 = \ln x^2 + \ln 3 = \ln\left(3x^2\right)$$

(ii)   Hence, given that x satisfies the equation $2\ln x + \ln 3 = \ln(5x + 2)$.
Show that x is a root of the quadratic equation $3x^2 - 5x - 2 = 0$

$$2\ln x + \ln 3 = \ln(5x + 2)$$
$$\ln x^2 + \ln 3 = \ln(5x + 2)$$
$$\ln\left(3x^2\right) = \ln(5x + 2)$$
$$\Rightarrow 3x^2 = 5x + 2 \quad \Rightarrow 3x^2 - 5x - 2 = 0$$

(iii)   Solve this quadratic equation, explaining why only one root is a valid solution of $2\ln x + \ln 3 = \ln(5x + 2)$

$$3x^2 = 5x + 2 \quad \Rightarrow 3x^2 - 5x - 2 = 0 \quad (3x + 1)(x - 2) = 0 \quad x = 2, \text{ and } x = -\frac{1}{3}$$

Here only one solution (i.e. x=2 is OK), the other is not valid because logarithm of negative numbers are not possible.

Ex-23-4:   Fig.-23-4 shows a cone. The angle between the axis and the slant edge is 30^0. Water is poured into the cone at a constant rate of $2\,cm^3$ per second. At time t seconds, the radius of the water surface is r cm and the volume of the water in the cone is $V\,cm^3$.
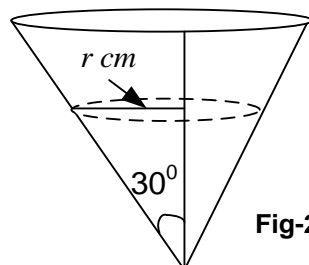
Fig-23-4

(i)   Write down the value of $\dfrac{dV}{dt}$

$$\frac{dV}{dt} = 2\frac{cm^3}{s}$$

(ii) Show that $V = \dfrac{\sqrt{3}}{3}\pi r^3$ and find $\dfrac{dV}{dr}$

You may assume that the volume of a cone of height h and radius r is $\dfrac{1}{3}\pi r^2 h$

$$\tan 30^0 = \frac{r}{h} \quad \Rightarrow h = \frac{r}{\tan 30^0}$$
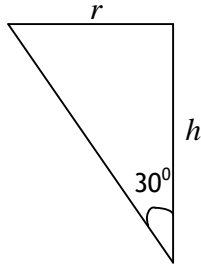
$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\left(\frac{r}{\tan 30^0}\right) = \frac{\sqrt{3}}{3}\pi r^3$$

$$\frac{dV}{dr} = \frac{3\sqrt{3}}{3}\pi r^2 = \sqrt{3}\,\pi r^2$$

(iii) Use the results of parts (i) and (ii) to find the value of $\dfrac{dr}{dt}$ when r=2.

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} \quad \Rightarrow 2 = \sqrt{3}\pi r^2 \times \frac{dr}{dt} \quad \Rightarrow \frac{dr}{dt} = \frac{2}{\sqrt{3}\pi r^2} = \frac{2}{4\sqrt{3}\pi} = \frac{1}{2\sqrt{3}\pi}$$

Ex-23-5: A curve is defined implicitly by the equation

$$y^3 = 2xy + x^2$$

(i) Show that $\dfrac{dy}{dx} = \dfrac{2(x+y)}{3y^2 - 2x}$

$$y^3 = 2xy + x^2$$

$$3y^2\frac{dy}{dx} = 2x\frac{dy}{dx} + 2y + 2x \quad \Rightarrow (3y^2 - 2x)\frac{dy}{dx} = 2y + 2x \quad \Rightarrow \frac{dy}{dx} = \frac{2(x+y)}{(3y^2 - 2x)}$$

(ii) Hence write down $\dfrac{dx}{dy}$ in terms of x and y.

$$\frac{dy}{dx} = \frac{2(x+y)}{(3y^2 - 2x)} \quad \Rightarrow \frac{dx}{dy} = \frac{3y^2 - 2x}{2(x+y)}$$

Ex-23-6: The function f(x) is defined by $f(x) = 1 + 2\sin x$ for $-\dfrac{1}{2}\pi \le x \le \dfrac{1}{2}\pi$.

(i)   Show that $f^{-1}(x) = \arcsin\left(\dfrac{x-1}{2}\right)$ and state the domain of this function.

$f(x) = 1 + 2\sin x$

$y = 1 + 2\sin x \Rightarrow \sin x = \dfrac{y-1}{2} \Rightarrow x = \arcsin\left(\dfrac{y-1}{2}\right)$

$f^{-1}(y) = \arcsin\left(\dfrac{y-1}{2}\right)$

$f^{-1}(x) = \arcsin\left(\dfrac{x-1}{2}\right)$

For domain $\left|\dfrac{x-1}{2}\right| \le 1 \Rightarrow \pm\left(\dfrac{x-1}{2}\right) \le 1$

$\left(\dfrac{x-1}{2}\right) \le 1 \Rightarrow x - 1 \le 2 \Rightarrow x \le 3$

$-\left(\dfrac{x-1}{2}\right) \le 1 \Rightarrow -(x-1) \le 2 \Rightarrow -x + 1 \le 2 \Rightarrow -x \le 1 \Rightarrow x \ge -1$

domain $-1 \le x \le 3$

Fig-23-6 shows a sketch of the graphs of $y = f(x)$ and $y = f^{-1}(x)$

(ii)   Write down the coordinates of the points A, B and C.
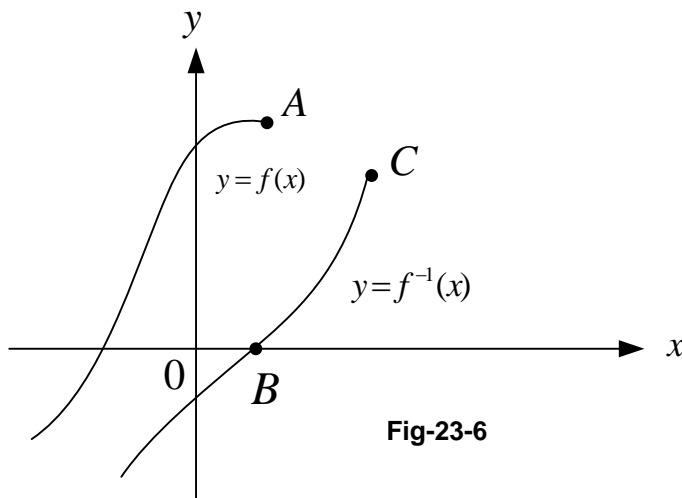
Fig-23-6

$y = f^{-1}(x) = \arcsin\left(\dfrac{x-1}{2}\right)$

$y\Big|_{at\ \mathbf{B}} = 0 \Rightarrow y = f^{-1}(x) = \arcsin\left(\dfrac{x-1}{2}\right) = 0 \Rightarrow \dfrac{x-1}{2} = \sin 0^{0} = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$

$B = (1,0)$

$y = f(x) = 1 + 2\sin x$

$y\Big|_{\max\ at\ A} \Rightarrow 1 + 2\sin x = 1 + 2 = 3 \ and \ x = \dfrac{\pi}{2} \Rightarrow A\left(\dfrac{\pi}{2}, 3\right)$

And point C is reflection of A: $C\left(3, \dfrac{\pi}{2}\right)$

Ex-23-7: Fig-23-7 shows the curve $y = 2x - x\ln x$, where $x > 0$

Fig-23-7
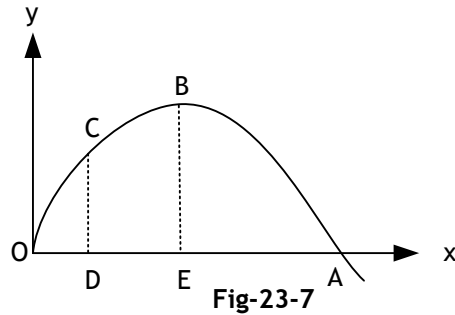
The curve crosses the x-axis at A, and has a turning point at B. The point C on the curve has x-coordinate 1. Lines CD and BE are drawn parallel to the y-axis.

(i) Find the x-coordinate of A, giving your answer in terms of e

$At \ A \ y = 0 \ \Rightarrow 2x - x\ln x = 0 \quad \Rightarrow x(2 - \ln x) = 0 \ \Rightarrow x = 0 \ (Po\text{int} \ O)$

$(2 - \ln x) = 0 \ \Rightarrow \ln x = 2 \ \Rightarrow x = e^2 \quad Po\text{int} \ A(e^2, 0)$

(ii) Find the exact coordinates of B

$y = 2x - x\ln x \quad \dfrac{dy}{dx} = 2 - x\left(\dfrac{1}{x}\right) - \ln x = 0 \quad \Rightarrow \ln x = 1 \ \Rightarrow x = e$

$y = 2x - x\ln x = 2e - e\ln e = 2e - e = e \qquad B(e, e)$

(iii) Show that the tangents at A and C are perpendicular to each other

$y = 2x - x\ln x \quad \dfrac{dy}{dx}\Big|_{at \ A} = 2 - x\left(\dfrac{1}{x}\right) - \ln x = 1 - \ln x\Big|_{x = e^2} = 1 - \ln e^2 = 1 - 2\ln e = 1 - 2 = -1$

$y = 2x - x\ln x \quad \dfrac{dy}{dx}\Big|_{at \ C} = 2 - x\left(\dfrac{1}{x}\right) - \ln x = 1 - \ln x\Big|_{x = 1} = 1 - \ln 1 = 1 - 0 = 1$

$Gradient \ at \ A = -\dfrac{1}{gradient \ at \ C} \ \Rightarrow Perpendicular$

(iv) Using integration by parts, show that

$$\int x \ln x \, dx = \frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 + c$$

$$\int x \ln x \, dx = \int u \, dv = uv - \int v \, du \quad \text{int } egration \quad by \quad part.$$

$$Let \quad u = \ln x \quad and \quad dv = x \, dx$$

$$\frac{du}{dx} = \frac{1}{x} \quad \Rightarrow du = \frac{1}{x} dx \quad and \quad v = \int x \, dx = \frac{x^2}{2}$$

$$\int x \ln x \, dx = \int u \, dv = uv - \int v \, du = \frac{x^2}{2} \ln x - \int \frac{x^2}{2x} dx = \frac{x^2}{2} \ln x - \int \frac{x}{2} dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + c$$

Hence, find the exact area of the region enclosed by the curve, the x-axis and the line CD and BE.

$$\int_{1}^{e} x \ln x \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} \Big|_{1}^{e} = \frac{e^2}{2} \ln e - \frac{e^2}{4} - \left(\frac{1}{2} \ln 1 - \frac{1}{4} \right) = \frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4}$$

Ex-23-8 The function $f(x) = \dfrac{\sin x}{2 - \cos x}$ has domain $-\pi \le x \le \pi$,

Fig-23-8 shows the graph of $y = f(x)$ for $0 \le x \le \pi$
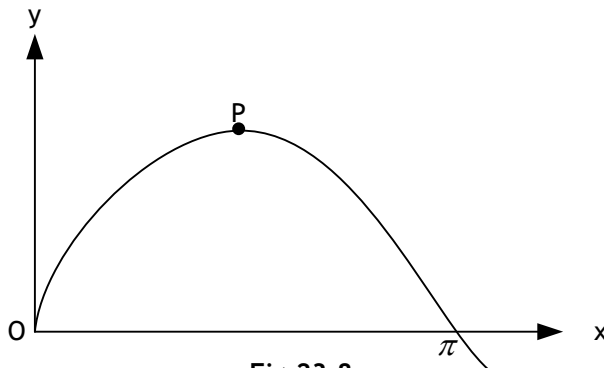
Fig-23-8

(i) Find $f(-x)$ in terms of $f(x)$. Hence sketch the graph of $y = f(x)$ for the complete domain $-\pi \le x \le \pi$

$$f(x) = \frac{\sin x}{2 - \cos x}$$

$$f(-x) = \frac{\sin(-x)}{2 - \cos(-x)} = \frac{-\sin x}{2 - \cos x} = -\frac{\sin x}{2 - \cos x} = -f(x) \quad \Rightarrow odd$$

(ii) Show that $f'(x) = \dfrac{2\cos x - 1}{(2 - \cos x)^2}$. Hence find the exact coordinates of the turning point P. State the range of the function $f(x)$, giving your answer exactly.

$$f(x) = \frac{\sin x}{2 - \cos x}$$

$$f'(x) = \frac{(2 - \cos x)\cos x - \sin x \sin x}{(2 - \cos x)^2} = \frac{2\cos x - \cos^2 x - \sin^2 x}{(2 - \cos x)^2} = \frac{2\cos x - (\cos^2 x + \sin^2 x)}{(2 - \cos x)^2}$$

$$\Rightarrow f'(x) = \frac{2\cos x - 1}{(2 - \cos x)^2}$$

$$\Rightarrow f'(x) = \frac{2\cos x - 1}{(2 - \cos x)^2} = 0 \quad \Rightarrow 2\cos x - 1 = 0 \quad \Rightarrow \cos x = \frac{1}{2} \quad \Rightarrow x = \frac{\pi}{3}$$

$$y = f(x) = \left.\frac{\sin x}{2 - \cos x}\right|_{x = \frac{\pi}{3}} = \frac{\sin\left(\frac{\pi}{3}\right)}{2 - \cos\left(\frac{\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{2 - 0.5} = \frac{\sqrt{3}}{3}$$

$$\left.y\right|_{\min} = \frac{\sin x}{2 - \cos x} = \frac{0}{2 - 1} = 0 \quad and \quad \left.y\right|_{\max} = \frac{1}{2 - 1} = \frac{1}{1} = 1 \quad \Rightarrow range \ \ 0 \le y \le 1$$

(iii) Using the substitution $u = 2 - \cos x$ or otherwise, find the exact value of

$$\int_0^\pi \frac{\sin x}{2 - \cos x}\,dx$$

$$u = 2 - \cos x \quad \Rightarrow \frac{du}{dx} = \sin x \quad \Rightarrow dx = \frac{du}{\sin x}$$

$$\int_0^\pi \frac{\sin x}{2 - \cos x}\,dx = \int \frac{\sin x}{u}\frac{du}{\sin x} = \int \frac{du}{u} = \ln u = \ln\left.(2 - \cos x)\right|_0^\pi = \ln(2 - \cos \pi) - \ln(2 - 1)$$

$$\int_0^\pi \frac{\sin x}{2 - \cos x} = \ln(2 - \cos \pi) - \ln(2 - 1) = \ln(2 + 1) - \ln 1 = \ln 3$$

(iv) Sketch the graph of $y = f(2x)$

(v) Using your answer to parts (iii) and (iv), write down the exact value of

$$\int_0^\pi \frac{\sin 2x}{2 - \cos 2x}\,dx$$

Area is stretched with scale factor $\frac{1}{2}$. So area is $\frac{1}{2}\ln 3$

PAPER-24

Ex-24-1: Solve the equation $|3x-2|=x$.

$$|3x-2|=x$$

$$(3x-2)=\pm x$$

(a) $(3x-2)=x \quad \Rightarrow 3x-x=2 \quad \Rightarrow 2x=2 \quad \Rightarrow x=1$

(b) $(3x-2)=-x \quad \Rightarrow 3x-2=-x \quad \Rightarrow 4x=2 \quad \Rightarrow x=\dfrac{1}{2}$

ans: $x=1 \quad and \quad x=\dfrac{1}{2}$

Ex-24-2: Show that $\displaystyle\int_{0}^{\frac{\pi}{6}} x\sin 2x\,dx = \dfrac{3\sqrt{3}-\pi}{24}$

$$\int_{0}^{\frac{\pi}{6}} x\sin 2x\,dx = ?$$

$$\int_{0}^{\frac{\pi}{6}} x\sin 2x\,dx = \int u\,dv = uv - \int v\,du \quad int egration \ \ by \ \ part.$$

$$Let \quad u=x \quad and \quad dv=\sin 2x\,dx$$

$$du=dx \quad and \quad v=\int \sin 2x\,dx = -\frac{1}{2}\cos 2x$$

$$\int_{0}^{\frac{\pi}{6}} x\sin 2x\,dx = \int u\,dv = uv - \int v\,du = -\frac{x}{2}\cos 2x + \int \frac{1}{2}\cos 2x\,dx = -\frac{x}{2}\cos 2x \Big|_{0}^{\frac{\pi}{6}} + \frac{1}{4}\sin 2x \Big|_{0}^{\frac{\pi}{6}}$$

$$\int_0^{\frac{\pi}{6}} x\sin 2x\,dx = -\frac{x}{2}\cos 2x\Bigg|_0^{\frac{\pi}{6}} + \frac{1}{4}\sin 2x\Bigg|_0^{\frac{\pi}{6}} = -\frac{\pi}{12}\cos 2\left(\frac{\pi}{6}\right) - 0 + \frac{1}{4}\sin 2\left(\frac{\pi}{6}\right) - 0 = -\frac{\pi}{12}\cos\left(\frac{\pi}{3}\right) + \frac{1}{4}\sin\left(\frac{\pi}{3}\right)$$

$$\int_0^{\frac{\pi}{6}} x\sin 2x\,dx = -\frac{\pi}{12}\cos\left(\frac{\pi}{3}\right) + \frac{1}{4}\sin\left(\frac{\pi}{3}\right) = -\frac{\pi}{12}\times\frac{1}{2} + \frac{1}{4}\times\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8} - \frac{\pi}{24} = \frac{3\sqrt{3} - \pi}{24}$$

Ex-24-3: Fig-24-3 shows the curve defined by the equation $y = \arcsin(x-1)$, for $0 \le x \le 2$.
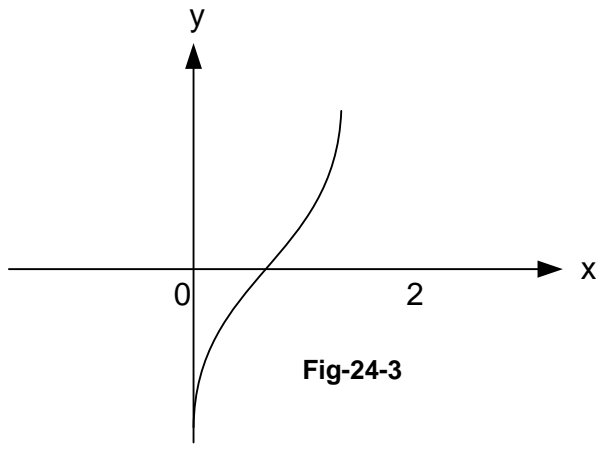
Fig-24-3

(i) Find x in terms of y, and show that $\dfrac{dx}{dy} = \cos y$.

$y = \arcsin(x-1) \Rightarrow x - 1 = \sin y \Rightarrow x = \sin y + 1$

$x = \sin y + 1$

$1 = \cos y\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cos y} \Rightarrow \dfrac{dx}{dy} = \cos y$

(ii) Hence find the exact gradient of the curve at the point where of $x = 1.5$.

$y = \arcsin(x-1)\Big|_{x=1.5} = \arcsin(1.5 - 1) = \arcsin(0.5) = 30^0$

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$$gradient = \frac{dy}{dx} = \frac{1}{\cos y}\bigg|_{y=30^0} = \frac{1}{\cos 30^0} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

Ex-24-4:   Fig-24-4 is a diagram of a garden pond.

h m
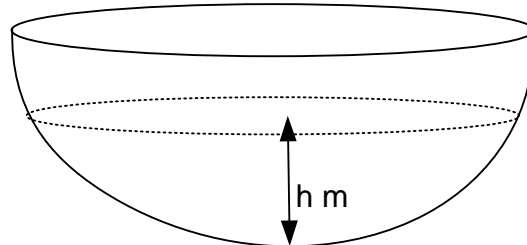
Fig-24-4

The volume $V\, m^3$ of water in the pond when the depth is h meters is given by

$$V = \frac{1}{3}\pi h^2 (3-h)$$

(i)   Find $\dfrac{dV}{dh}$

$$V = \frac{1}{3}\pi h^2 (3-h) = \pi h^2 - \frac{1}{3}\pi h^3$$

$$\frac{dV}{dh} = 2\pi h - \pi h^2$$

Water is poured into the pond at the rate of $0.02\, m^3$ per minute

(ii)   Find the value of $\dfrac{dh}{dt}$ when $h = 0.4$

$$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV} = 0.02 \left(\frac{1}{2\pi h - \pi h^2}\right)\bigg|_{h=0.4} = 0.02\left(\frac{1}{2\pi(0.4)-\pi(0.4)^2}\right) = \frac{0.02}{2.011} = 0.01$$

Ex-24-5:   Positive integers a, b and c are said to form a Pythagorean triple if $a^2 + b^2 = c^2$

(i)   Given that t is an integer greater than 1, show that $2t$, $t^2 - 1$, $t^2 + 1$ form a Pythagorean triple.

$$c = t^2 + 1$$
$$a = 2t$$
$$90^0$$
$$b = t^2 - 1$$
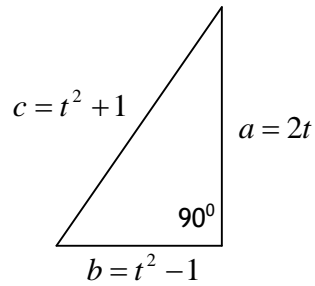
$$a^2 + b^2 = c^2 \Rightarrow (2t)^2 + (t^2 - 1)^2 = \left(t^2 + 1\right)^2$$

$$\Rightarrow 4t^2 + t^4 - 2t^2 + 1 = t^4 + 2t^2 + 1$$

$$\Rightarrow t^4 + 2t^2 + 1 = t^4 + 2t^2 + 1 \quad \text{hence, OK}$$

(ii)   The two smallest integers of a Pythagorean triple are 20 and 21. Find the third integer.

$$c^2 = a^2 + b^2 = (20)^2 + (21)^2 = 400 + 441 = 841 \quad \Rightarrow c = \sqrt{841} = 29$$

Use this triple to show that not all Pythagorean triples can be expressed in the form $2t$, $t^2 - 1$ and $t^2 + 1$

Let $2t = 20 \Rightarrow t = 10$ and $t^2 - 1 = (10)^2 - 1 = 99$ and $t^2 + 1 = (10)^2 + 1 = 101$

Hence $(20)^2 + (99)^2 \neq (101)^2$ and therefore not valid for every t

Ex-24-6:   The mass $M \ kg$ of a radioactive material is modelled by the equation

$$M = M_0 e^{-kt}$$

Where M_0 is the initial mass, t is the time in years, and k is a constant which measures the rate of radioactive decay.

(i)   Sketch the graph of M against t
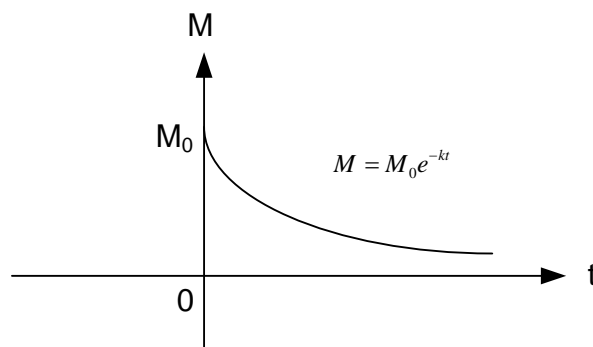
(ii)   For carbon 14, $k = 0.000121$. Verify that after 5730 years the mass M has reduced to approximately half the initial mass.

$$M = M_0 e^{-kt}\Big|_{t=5730} = M_0 e^{-0.000121(5730)} = 0.4999\, M_0 \cong \frac{1}{2} M_0$$

The half-life of a radioactive material is the time taken for its mass to reduce to exactly half the initial mass.

(iii)   Show that, in general, the half-life T is given by

$$\frac{1}{2} M_0 = M_0 e^{-kt} \Rightarrow e^{-kt} = \frac{1}{2} \Rightarrow -kt \ln e = -kt = \ln\left(\frac{1}{2}\right) = \ln 1 - \ln 2 = -\ln 2$$

$$-kt = -\ln 2 \quad \Rightarrow t = \frac{\ln 2}{k}$$

(iv)   Hence find the half-life of Plutonium 239, given that for this material $k = 2.88 \times 10^{-5}$

$$\Rightarrow t = \frac{\ln 2}{k} = \frac{0.693}{2.88 \times 10^{-5}} = 24068$$

SECTION B (P-24)

Ex-24-7:   Fig-24-7 shows the curve $y = \dfrac{x^2 + 3}{x - 1}$. It has a minimum at the point P.

The line L is an asymptote to the curve.

Fig-24-7
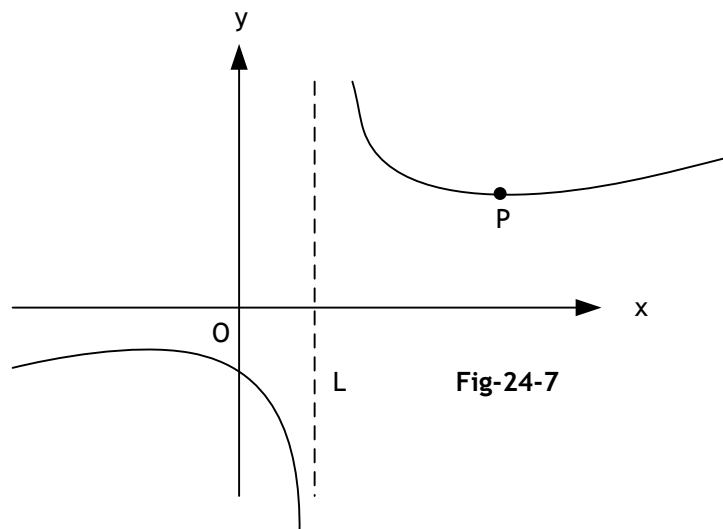
(i)   Write down the equation of the asymptote L.

$$y = \frac{x^2 + 3}{x - 1}$$

Asymptote is the value of x which makes y equal to ∞

$$x - 1 = 0 \implies x = 1 \ is \ asymptote$$

(ii)   Find the coordinates of P.

$$y = \frac{x^2 + 3}{x - 1}$$

$$\implies \frac{dy}{dx} = \frac{(x-1)(2x) - (x^2 + 3)(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 3}{(x-1)^2} = \frac{x^2 - 2x - 3}{(x-1)^2} = 0$$

$$x^2 - 2x - 3 = 0 \implies (x+1)(x-3) = 0 \implies x = -1 \ and \ x = 3 \implies x = 3 \ is \ OK$$

$$y = \frac{x^2 + 3}{x-1}\bigg|_{x=3} = \frac{9+3}{3-1} = \frac{12}{2} = 6 \implies Po\,int = (3,6)$$

(iii)   Using the substitution $u = x - 1$, show that the area of the region enclosed by the x-axis, the curve and the lines $x = 2$ and $x = 3$ is given by

$$\int_2^3 \frac{x^2 + 3}{x - 1} dx = \int \left(\frac{u^2 + 2u + 4}{u} \right) du = \int_1^2 \left(u + 2 + \frac{4}{u} \right) du$$

Evaluate this area exactly.

Fig-24-7
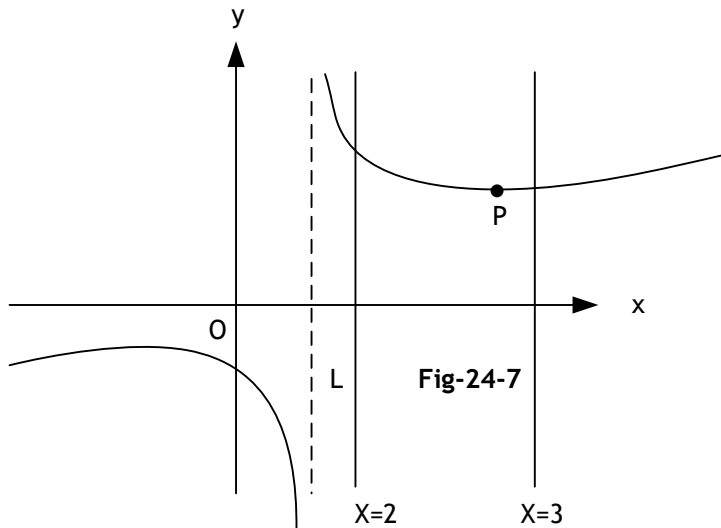
$$\int_2^3 \frac{x^2 + 3}{x - 1} dx$$

$$u = x - 1 \implies \frac{du}{dx} = 1 \implies dx = du$$

$$u = x - 1 \implies x = u + 1 \implies x^2 = (u+1)^2 = u^2 + 2u + 1 \implies x^2 + 3 = u^2 + 2u + 4$$

$$u = x - 1 \implies u = 3 - 1 = 2 \quad and \quad u = 2 - 1 = 1$$

$$\int_2^3 \frac{x^2 + 3}{x - 1} dx = \int_1^2 \left(\frac{u^2 + 2u + 4}{u} \right) du$$

$$= \int_1^2 \left(u + 2 + \frac{4}{u} \right) du = \frac{u^2}{2} + 2u + 4 \ln u \Big|_1^2 = \frac{4}{2} + 4 + 4 \ln 2 - \left[\frac{1}{2} + 2 + 4 \ln 1 \right] = 6 + 4 \ln 2 - \frac{1}{2} - 2 - 4(0) = \frac{7}{2} + 4 \ln 2$$

(iv)   Another curve is defined by the equation $e^y = \frac{x^2 + 3}{x - 1}$. Find $\frac{dy}{dx}$ In terms of x and y by differentiating implicitly. Hence find the gradient of this curve at the point where $x = 2$

$$e^y = \frac{x^2 + 3}{x - 1}$$

$$e^y \frac{dy}{dx} = \frac{(x-1)(2x) - (x^2 + 3)(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 3}{(x-1)^2} = \frac{x^2 - 2x - 3}{(x-1)^2}$$

$$\implies \frac{dy}{dx} = \frac{x^2 - 2x - 3}{e^y (x-1)^2}$$

$$e^y = \frac{x^2 + 3}{x - 1} \Big|_{x=2} = \frac{4 + 3}{2 - 1} = 7$$

$$\implies \frac{dy}{dx} = \frac{x^2 - 2x - 3}{e^y (x-1)^2} \Big|_{x=2} = \frac{4 - 4 - 3}{7} = -\frac{3}{7}$$

Ex-24-8:   Fig-24-8 shows part of the curve $y = f(x)$, where $f(x) = e^{-\frac{1}{5}x} \sin x$ for all x

Fig-24-8
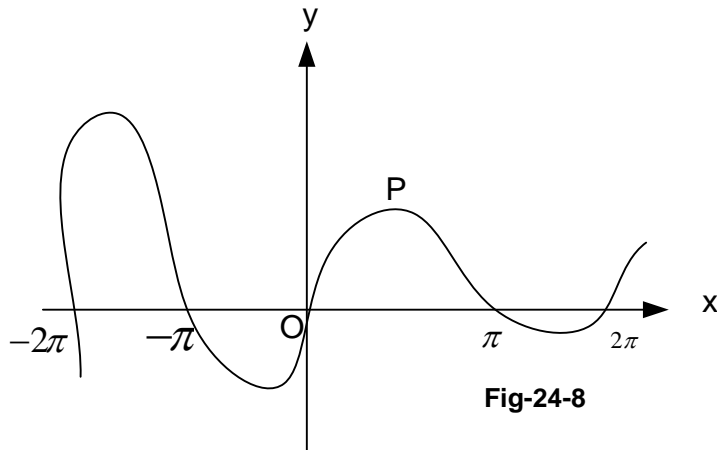
(i) Sketch the graph of

(A) $y = f(2x)$

(B) $y = f(x + \pi)$

(ii) Show that the x-coordinate of the turning point P satisfies the equation $\tan x = 5$

Hence find the coordinates of P

$$y = f(x) = e^{-\frac{1}{5}x} \sin x$$

$$\frac{dy}{dx} = e^{-\frac{1}{5}x}(\cos x) - \frac{1}{5}e^{-\frac{1}{5}x}\sin x = 0$$

$$(\cos x) - \frac{1}{5}\sin x = 0 \implies \sin x = 5\cos x \implies \frac{\sin x}{\cos x} = \tan x = 5$$

$$\tan x = 5 \implies x = \tan^{-1}(5) = 87.43^{0} = 1.37 \ rad$$

$$y = e^{-\frac{1}{5}x}\sin x\Big|_{x = 1.37 \ rad} = e^{-\frac{1.37}{5}}\sin(1.37) = 0.76(0.9799) = 0.744$$

P(1.37 rad, 0.744)

(iii) Show that $f(x + \pi) = -e^{-\frac{1}{5}\pi}f(x)$. Hence, using the substitution $u = x - \pi$, show

that $\displaystyle\int_{\pi}^{2\pi} f(x)dx = -e^{-\frac{1}{5}\pi}\int_{0}^{\pi} f(u)du$

$$y = f(x) = e^{-\frac{1}{5}x}\sin x$$

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$$f(x+\pi) = e^{-\frac{x+\pi}{5}} \sin(x+\pi) = e^{-\frac{\pi}{5}} e^{-\frac{x}{5}}(-\sin x) = -e^{-\frac{\pi}{5}} e^{-\frac{x}{5}} \sin x = -e^{-\frac{\pi}{5}} f(x)$$

Interpret this result graphically. [You should not attempt to integrate $f(x)$]

PAPER-25

SECTION A (C3-18-1-07)

Ex-25-1: Fig-25-1 shows the graph of $y=|x|$ and $y=|x-2|+1$. The point P is the minimum point of $y=|x-2|+1$ and Q is the point of intersection of the two graphs.

Fig-25-1
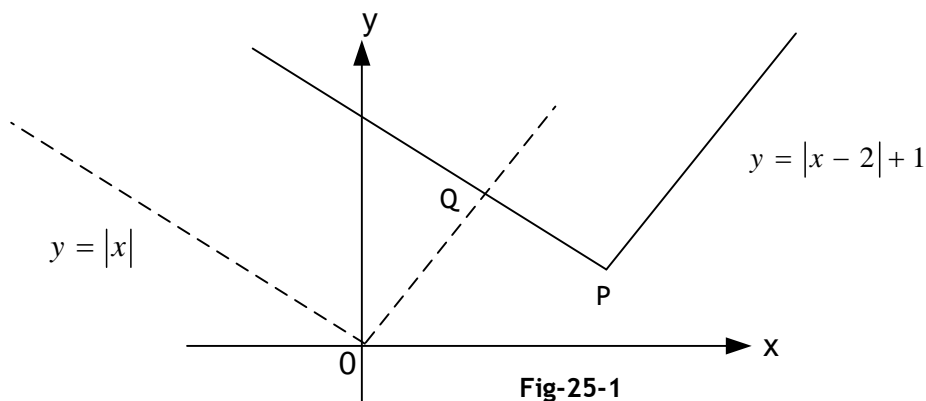
(i)   Write down the coordinates of P.

$$y=|x-2|+1$$

y is minimum when $|x-2|=0 \Rightarrow x=2$

$$y=|x-2|+1=1 \ at \ x=2 \Rightarrow P(2,1)$$

(ii)   Verify that the y-coordinate of Q is $1\frac{1}{2}$

At point Q, $y_1=y_2$ and also $y=|x|=x$ and $y=|x-2|+1=-(x-2)+1=3-x$ lso

$$x=3-x \Rightarrow 2x=3 \Rightarrow x=1\frac{1}{2} \quad and \quad y=x=1\frac{1}{2}$$

Ex-25-2 Evaluate $\int\limits_1^2 x^2 \ln x\,dx$, giving your answer in an exact form.

$$\int\limits_1^2 x^2 \ln x\,dx = \int u\,dv = uv - \int v\,du \quad \text{int } egration \ by \ part.$$

$$Let \quad u = \ln x \quad and \quad dv = x^2 dx$$

$$du = \frac{dx}{x} \quad and \quad v = \int x^2 dx = \frac{1}{3}x^3$$

$$\int\limits_1^2 x^2 \ln x\,dx = \int u\,dv = uv - \int v\,du = \frac{x^3}{3}\ln x - \int \frac{1}{3}x^3\frac{dx}{x} = \frac{x^3}{3}\ln x - \int \frac{1}{3}x^2 dx = \frac{x^3}{3}\ln x\Bigg|_1^2 - \frac{1}{9}x^3\Bigg|_1^2 .$$

$$\int\limits_1^2 x^2 \ln x\,dx = \frac{x^3}{3}\ln x\Bigg|_1^2 - \frac{1}{9}x^3\Bigg|_1^2 = \frac{8}{3}\ln 2 - \frac{1}{3}\ln 1 - \frac{1}{9}(8-1) = \frac{8}{3}\ln 2 - \frac{7}{9}$$

Ex-25-3 The value V of a car is modeled by the equation of $V = Ae^{-kt}$ where t is the age of the car in years and A and k are constants. Its value when new is of 10000 , and after 3 years its value is of 6000

(i) Find the values of A and k.

$$V = Ae^{-kt}\Bigg|_{t=0} = A = 10000$$

$$V = 10000\,e^{-kt}\Bigg|_{t=3} = 6000 \Rightarrow 10000\,e^{-3k} = 6000$$

$$\Rightarrow e^{-3k} = \frac{6000}{10000} = 0.6 \quad \Rightarrow -3k\ln e = -3k = \ln(0.6) = -0.511 \quad \Rightarrow k = \frac{-0.511}{-3} = 0.17$$

(ii) Find the age of the car when its value is 2000

$$V = 10000\,e^{-0.17t} = 2000 \quad \Rightarrow e^{-0.17t} = \frac{2000}{10000} = 0.2 \quad \Rightarrow -0.17t = \ln(0.2) \Rightarrow t = \frac{\ln(0.2)}{-0.17}$$

$$\Rightarrow t = \frac{\ln(0.2)}{-0.17} = 9.4673 \ years$$

Ex-25-4 Use the method of exhaustion to prove the following result.

No 1- or 2-digit perfect square ends in 2, 3, 7 or 8.

Ans: 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, ...

State a generalization of this result.

Ex-25-5   The equation of a curve is $y = \dfrac{x^2}{2x+1}$

(i)   Show that $\dfrac{dy}{dx} = \dfrac{2x(x+1)}{(2x+1)^2}$

$$y = \frac{x^2}{2x+1} \Rightarrow \frac{dy}{dx} = \frac{(2x+1)(2x) - x^2(2)}{(2x+1)^2} = \frac{4x^2 + 2x - 2x^2}{(2x+1)^2} = \frac{2x^2 + 2x}{(2x+1)^2} = \frac{2x(x+1)}{(2x+1)^2}$$

(ii)   Find the coordinates of the stationary points of the curve. You need not determine their nature.

$$\frac{dy}{dx} = \frac{2x(x+1)}{(2x+1)^2} = 0 \quad \Rightarrow 2x(x+1) = 0 \quad \Rightarrow x = 0, \ \ x = -1$$

$$y = \frac{x^2}{2x+1}\bigg|_{x=0} = 0 \quad \Rightarrow P(0,0) \quad and \quad y = \frac{x^2}{2x+1}\bigg|_{x=-1} = \frac{1}{-2+1} = -1 \quad \Rightarrow P(-1,-1)$$

Ex-25-6   Fig-25-6 shows the triangle OAP, where O is the origin and A is the point (0, 3). The point P(x, 0) moves on the positive x-axis. The point Q(0, y) moves between O and A in such a way that AQ+AP=6

Fig-25-6
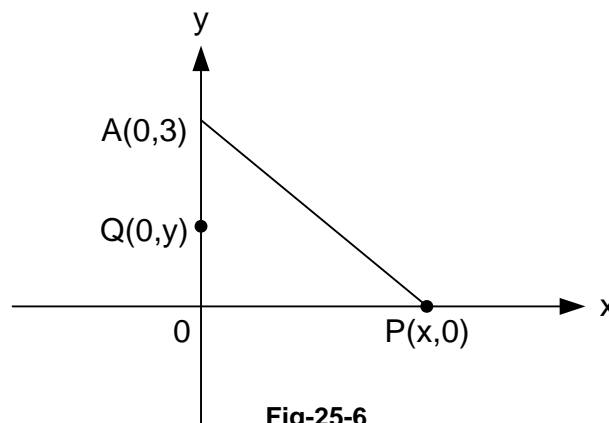
(i)   Write down the length AQ in terms of y. Hence find AP in terms of y, and show that

$$(y+3)^2 = x^2 + 9$$

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A(0,3)

Q(0,y)

0

P(x,0)

y

x

Fig-25-6
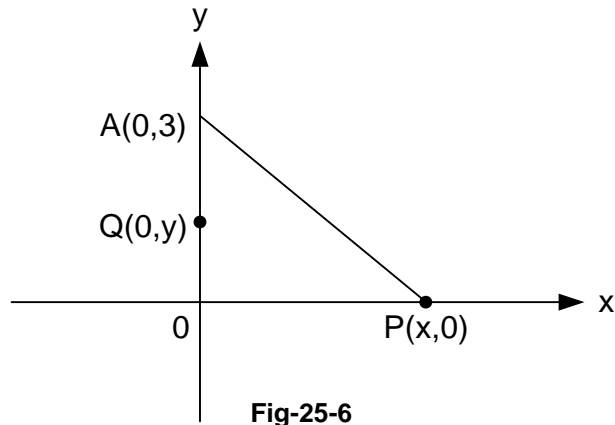
$AQ = (3 - y)$

$AP = \sqrt{(0-3)^2 + (x-0)^2} = \sqrt{9 + x^2}$

$AQ + AP = 6$

$AP = 6 - AQ = 6 - (3 - y) = 6 - 3 + y = 3 + y = \sqrt{9 + x^2} \Rightarrow (y+3)^2 = 9 + x^2$

(ii)   Use this result to show that $\dfrac{dy}{dx} = \dfrac{x}{y+3}$

$(y+3)^2 = 9 + x^2$

$2(y+3)\dfrac{dy}{dx} = 2x \Rightarrow \dfrac{dy}{dx} = \dfrac{2x}{2(y+3)} = \dfrac{x}{y+3}$

(iii)   When x=4 and y=2, $\dfrac{dx}{dt} = 2$. Calculate $\dfrac{dy}{dt}$ at this time.

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt}$

$\Rightarrow \dfrac{dy}{dx}\bigg|_{x=4,\ y=2} = \dfrac{x}{y+3} = \dfrac{4}{2+3} = \dfrac{4}{5}$

$\dfrac{dy}{dt} = \dfrac{dy}{dx} \times \dfrac{dx}{dt} = \left(\dfrac{4}{5}\right)(2) = \dfrac{8}{5}$

SECTION B(P-25)

Ex-25-7   Fig-25-7 shows part of the curve $y = f(x)$, where $f(x) = x\sqrt{1+x}$. The curve meets the x-axis at the origin and at the point P.

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Fig-25-7
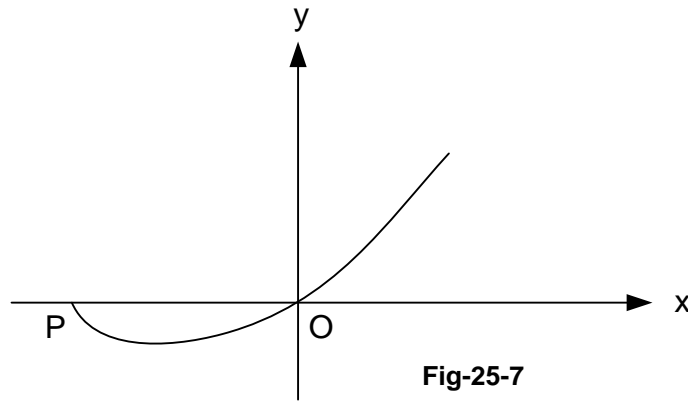
(i) Verify that the point P has coordinates (-1, 0). Hence state the domain of the function $f(x)$

$$y = f(x) = x\sqrt{1+x}$$

$$y = 0 \ \ at \ \ P \Rightarrow x\sqrt{1+x} = 0 \ \Rightarrow x = 0, \ and \ \ \sqrt{1+x} = 0 \ \Rightarrow 1+x = 0 \ \Rightarrow x = -1$$

Point P(-1, 0).

(ii) Show that $\dfrac{dy}{dx} = \dfrac{2+3x}{2\sqrt{1+x}}$

$$y = f(x) = x\sqrt{1+x}$$

$$y = f(x) = x(1+x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = x\left(\frac{1}{2}(1+x)^{-\frac{1}{2}}\right) + (1+x)^{\frac{1}{2}} = \frac{x}{2\sqrt{1+x}} + \frac{\sqrt{1+x}}{1} = \frac{x + 2\sqrt{1+x}\left(\sqrt{1+x}\right)}{2\sqrt{1+x}} = \frac{x + 2(1+x)}{2\sqrt{1+x}} = \frac{3x+2}{2\sqrt{1+x}}$$

(iii) Find the exact coordinates of the turning point of the curve. Hence write down the range of the function

$$\frac{dy}{dx} = \frac{3x+2}{2\sqrt{1+x}} = 0 \ \Rightarrow 3x+2 = 0 \ \ \Rightarrow x = -\frac{2}{3}$$

$$y = x\sqrt{1+x}\,\bigg|_{x=-\frac{2}{3}} = -\frac{2}{3}\sqrt{1-\frac{2}{3}} = -\frac{2}{3}\sqrt{\frac{3-2}{3}} = -\frac{2}{3}\sqrt{\frac{1}{3}} = -\frac{2}{3\sqrt{3}}$$

$$Po\,\text{int} \left(-\frac{2}{3}, -\frac{2}{3\sqrt{3}}\right)$$

(iv)   Use the substitution of $u = 1 + x$ to show that $\int\limits_{-1}^{0} x\sqrt{1+x}\,dx = \int\limits_{0}^{1}\left(u^{\frac{3}{2}} - u^{\frac{1}{2}} \right)du$

Hence find the area of the region enclosed by the curve and the x-axis.

$$\int\limits_{-1}^{0} x\sqrt{1+x}\,dx$$

$u = 1 + x \quad \Rightarrow \dfrac{du}{dx} = 1 \quad \Rightarrow dx = du$

$u = 1 + x \quad \Rightarrow x = u - 1 \quad$ limits for u=0, u=1

$$\int\limits_{-1}^{0} x\sqrt{1+x}\,dx = \int\limits_{0}^{1}(u-1)u^{\frac{1}{2}}\,du = \int\limits_{0}^{1}\left(u^{\frac{3}{2}} - u^{\frac{1}{2}} \right)du$$

Ex-25-8   Fig-25-8 shows part of the curve $y = f(x)$ where $f(x) = \left(e^{x} - 1\right)^{2}$ for $x \geq 0$

Fig-25-8
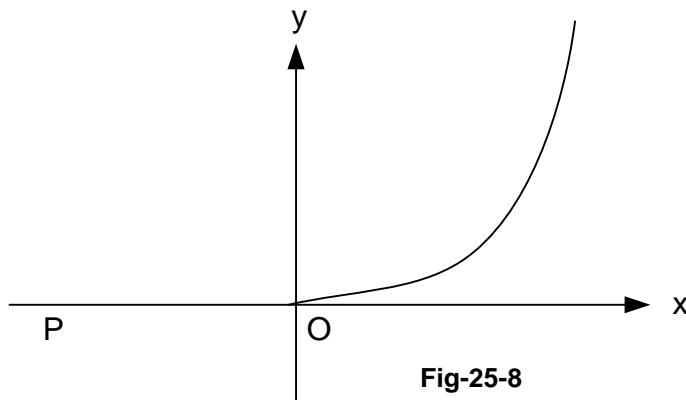
(i)   Find $f'(x)$, and hence calculate the gradient of the curve $y = f(x)$ at the origin and at the point (ln2, 1).

$f(x) = \left(e^{x} - 1\right)^{2}$

$\dfrac{dy}{dx} = 2\left(e^{x} - 1\right)e^{x} = 2(e^{2x} - e^{x})$

$gradient\Big|_{at\ origin} = \dfrac{dy}{dx}\Big|_{at\ origin} = 2(e^{2x} - e^{x})\Big|_{x=0} = 2(1-1) = 0$

$gradient\Big|_{x=\ln 2} = \dfrac{dy}{dx}\Big|_{x=\ln 2} = 2(e^{2x} - e^{x})\Big|_{x=\ln 2} = 2(e^{2\ln 2} - e^{\ln 2}) = 2(e^{\ln 2^{2}} - e^{\ln 2}) = 2(4-2) = 4$

The function $g(x)$ is defined by $g(x) = \ln\left(1 + \sqrt{x}\right)$ for $x \geq 0$

(ii) Show that f(x) and g(x) are inverse functions. Hence, sketch the graph of y=g(x). Write down the gradient of the curve y=g(x) at the point (1, ln2).

$$y = \left(e^x - 1\right)^2 \Rightarrow e^x - 1 = \sqrt{y} \Rightarrow e^x = \sqrt{y} + 1 \Rightarrow x = \ln(\sqrt{y} + 1)$$

$$f^{-1}(y) = \ln(\sqrt{y} + 1)$$

$$f^{-1}(x) = \ln(\sqrt{x} + 1) = g(x)$$

$$g(x) = \ln\left(1 + \sqrt{x}\right)$$

$$g'(x) = \left.\frac{1}{2\sqrt{x}\left(1 + \sqrt{x}\right)}\right|_{x=1} = \frac{1}{2(1+1)} = \frac{1}{4}$$

(iii) Show that $\int \left(e^x - 1\right)^2 dx = \frac{1}{2}e^{2x} - 2e^x + x + c$

$$\int \left(e^x - 1\right)^2 dx = \int (e^{2x} - 2e^x + 1)dx = \frac{1}{2}e^{2x} - 2e^x + x + c$$

Hence evaluate $\int_0^{\ln 2} \left(e^x - 1\right)^2 dx$, giving your answer in exact form.

$$\int_0^{\ln 2} \left(e^x - 1\right)^2 dx = \left.\frac{1}{2}e^{2x} - 2e^x + x\right|_0^{\ln 2} = \frac{1}{2}e^{2\ln 2} - 2e^{\ln 2} + \ln 2 - \left(\frac{1}{2} - 2\right) = 2 - 4 + \ln 2 + \frac{3}{2} = \ln 2 - \frac{1}{2}$$

(iv) Using your answer to part (iii), calculate the area of the region enclosed by the curve $y = g(x)$, the x-axis and the line $x = 1$

PAPER-26

SECTION A (C3-11-6-07)

Ex-26-1: **(i)** Differentiate $\sqrt{1 + 2x}$.

$$y = \sqrt{1 + 2x}$$

$$y = \left(1 + 2x\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(1+2x)^{-\frac{1}{2}}(2) = (1+2x)^{-\frac{1}{2}} = \frac{1}{\sqrt{1+2x}}$$

(ii)   Show that derivative of $\ln\left(1-e^{-x}\right)$ is $\dfrac{e^{-x}}{e^{x}-1}$

$$y = \ln\left(1-e^{-x}\right)$$

$$\frac{dy}{dx} = \frac{e^{-x}}{\left(1-e^{-x}\right)}$$

Ex-26-2:   Given that $f(x)=1-x$ and $g(x)=|x|$, write down the composite function $gf(x)$

$$gf(x) = g(1-x) = |1-x|$$

On separate diagrams, sketch the graphs of $y = f(x)$ and $y = gf(x)$

$$y = f(x) = 1-x \text{ and } y = gf(x) = |1-x|$$
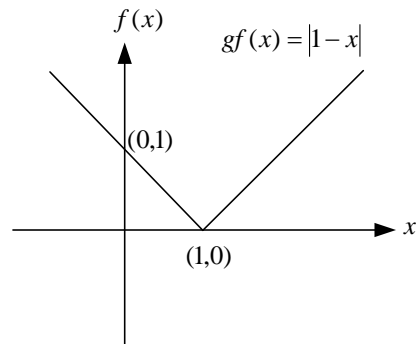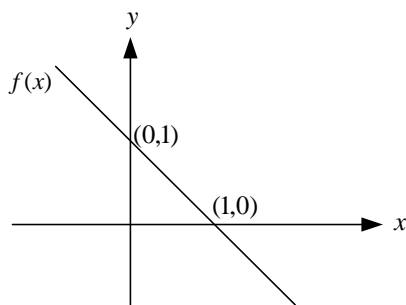
Ex-26-3:   A curve has equation $2y^{2} + y = 9x^{2} + 1$

(i)   Find $\dfrac{dy}{dx}$ in terms of x and y. Hence find the gradient of the curve at the point A(1,2)

$$2y^{2} + y = 9x^{2} + 1$$

$$4y\frac{dy}{dx} + \frac{dy}{dx} = 18x \quad \Rightarrow \frac{dy}{dx}(4y+1)=18x \quad \Rightarrow \frac{dy}{dx} = \frac{18x}{4y+1}$$

$$Gradient = \frac{dy}{dx} = \frac{18x}{4y+1}\bigg|_{x=1,\ y=2} = \frac{18}{8+1} = \frac{18}{9} = 2$$

(ii)   Find the coordinates of the points on the curve at which $\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{18x}{4y+1} = 0 \quad \Rightarrow 18x = 0 \ \Rightarrow x = 0$$

$$2y^2 + y = 0 + 1 \ \Rightarrow 2y^2 + y - 1 = 0 \ \Rightarrow (2y-1)(y+1) = 0 \ \Rightarrow y = \frac{1}{2}, \ y = -1$$

Points: P1(0,0.5) and P2(0,-1)

Ex-26-4:   A cup of water is cooling. Its initial temperature is 100^0C . After 3 minutes, its temperature is 80^0C

(i)   Given that $T = 25 + ae^{-kt}$ where T is the temperature in ^0C , t is the time in minutes and **a** and **k** are constants, find the values of **a** and **k**.

$$T = 25 + ae^{-kt}\bigg|_{t=0} = 25 + a = 100 \quad \Rightarrow a = 75$$

$$80 = 25 + 75e^{-3k} \ \Rightarrow 75e^{-3k} = 80 - 25 = 55 \ \Rightarrow e^{-3k} = \frac{55}{75} = \frac{11}{15}$$

$$\Rightarrow e^{-3k} = \frac{11}{15} \ \Rightarrow -3k = \ln\left(\frac{11}{15}\right) = -0.31 \ \Rightarrow k = \frac{-0.31}{-3} = 0.1$$

(ii)   What is the temperature of the water

(A)   After 5 minutes,

$$T = 25 + 75e^{-0.1t}\bigg|_{t=5} = 25 + 75e^{-0.5} = 25 + 75(0.606) = 70.49$$

(B)   in the long term?

$$T = 25 + 75e^{-0.1t}\bigg|_{t \to \infty} = 25 + 75e^{-\infty} = 25 + \frac{75}{e^{\infty}} = 25 + 0 = 25$$

Ex-26-5:   Prove that the following statement is false.

For all integers n greater than or equal to 1, $n^2 + 3n + 1$ is a prime number.

$n = 1 \Rightarrow n^2 + 3n + 1 = 1 + 3 + 1 = 5 \Rightarrow prime\ number$

$n = 2 \Rightarrow n^2 + 3n + 1 = 4 + 6 + 1 = 11 \Rightarrow prime\ number$

$n = 3 \Rightarrow n^2 + 3n + 1 = 9 + 9 + 1 = 19 \Rightarrow prime\ number$

$n = 4 \Rightarrow n^2 + 3n + 1 = 16 + 12 + 1 = 29 \Rightarrow prime\ number$

Ex-26-6:   Fig-26-6 shows the curve $y = f(x)$, where $f(x) = \dfrac{1}{2}\arctan x$
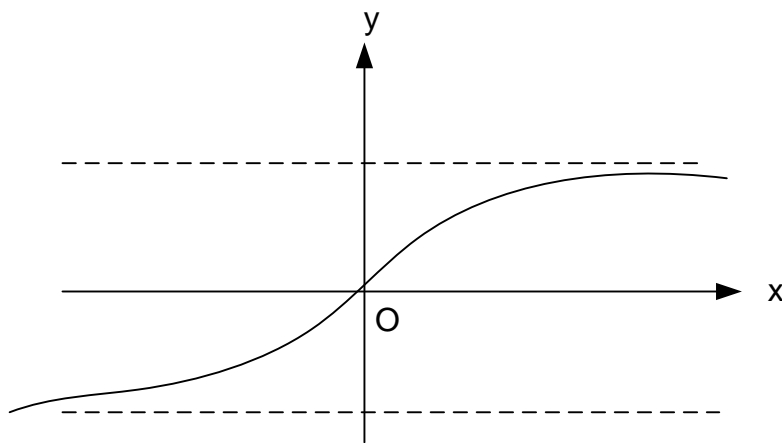
Fig-26-6

(i)   Find the range of the function $f(x)$, giving your answer in terms of π.

$$y = f(x) = \frac{1}{2}\arctan x \qquad x \to \infty \quad and \quad x \to -\infty$$

$$y = f(x) = \frac{1}{2}\arctan(\infty) = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

$$y = f(x) = \frac{1}{2}\arctan(-\infty) = \frac{1}{2}\left(-\frac{\pi}{2}\right) = -\frac{\pi}{4}$$

Therefore, range (i.e. value of y: $-\dfrac{\pi}{4} \le y \le \dfrac{\pi}{4}$)

(ii)   Find the inverse function $f^{-1}(x)$. Find the gradient of the curve $y = f^{-1}(x)$ at the origin.

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$$y = f(x) = \frac{1}{2}\arctan x \quad \Rightarrow x = \tan(2y)$$

$$f^{-1}(y) = \tan(2y)$$

$$f^{-1}(x) = \tan(2x)$$

$$f^{-1}(x) = g(x) = \tan(2x) = \frac{\sin 2x}{\cos 2x}$$

$$\frac{dg}{dx} = \frac{\cos 2x(2\cos 2x) - \sin 2x(-2\sin 2x)}{\cos^2 2x} = \frac{2}{\cos^2 2x}\bigg|_{x=0} = gradient = 2$$

(iii)   Hence write down the gradient of $y = \frac{1}{2}\arctan x$ at the origin.

$$y = \frac{1}{2}\arctan x \quad \Rightarrow x = \tan 2y \Rightarrow 1 = \frac{2}{\cos^2 2y}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{\cos^2 2y}{2}\bigg|_{y=0} = \frac{1}{2}$$

SECTION B (P-26)

Ex-26-7:   Fig-26-7 shows the curve $y = \dfrac{x^2}{1+2x^3}$. It is undefined at $x = a$; the line

   $x = a$ is a vertical asymptote.
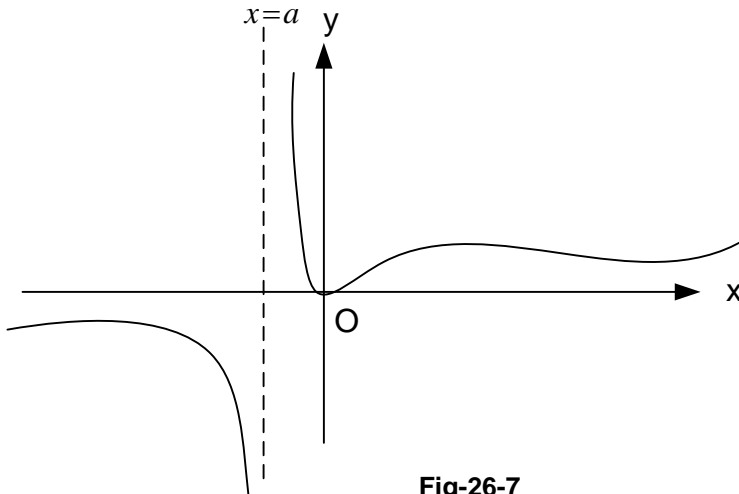
Fig-26-7

(i)   Calculate the value of a , giving your answer correct to 3 significant figures.

$$y = \frac{x^2}{1+2x^3} \quad for\ asymptote\ \ 1+2x^3 = 0 \ \Rightarrow x^3 = -\frac{1}{2} \ \Rightarrow x = -\frac{1}{\sqrt[3]{2}}$$

$$\Rightarrow x = -\frac{1}{\sqrt[3]{2}}$$

$$\Rightarrow x = a$$

$$\Rightarrow a = -\frac{1}{\sqrt[3]{2}}$$

(ii) Show that $\dfrac{dy}{dx} = \dfrac{2x - 2x^4}{\left(1 + 2x^3\right)^2}$. Hence determine the coordinates of the turning points of the curve.

$$y = \frac{x^2}{1 + 2x^3}$$

$$\frac{dy}{dx} = \frac{\left(1 + 2x^3\right)2x - x^2\left(6x^2\right)}{\left(1 + 2x^3\right)^2} = \frac{2x + 4x^4 - 6x^4}{\left(1 + 2x^3\right)^2} = \frac{2x - 2x^4}{\left(1 + 2x^3\right)^2}$$

$$\frac{dy}{dx} = \frac{2x - 2x^4}{\left(1 + 2x^3\right)^2} = 0 \Rightarrow 2x - 2x^4 = 2x(1 - x^3) = 0 \quad x = 0 \quad and \quad \left(1 - x^3\right) = 0 \Rightarrow x^3 = 1$$

$$\Rightarrow x^3 = 1 \quad \Rightarrow x = 1$$

$$y = \frac{x^2}{1 + 2x^3}\bigg|_{x=1} = \frac{1}{3} \quad \Rightarrow Po\operatorname{int}\left(1, \frac{1}{3}\right)$$

(iii) Show that the area of the region between the curve and the x-axis from $x = 0$ to $x = 1$ is $\dfrac{1}{6}\ln 3$

$$\int_0^1 \frac{x^2}{1 + 2x^3}\, dx$$

$$u = 1 + 2x^3 \quad \Rightarrow \frac{du}{dx} = 6x^2 \Rightarrow dx = \frac{du}{6x^2}$$

$$\int_0^1 \frac{x^2}{1 + 2x^3}\, dx = \int \frac{x^2}{u}\frac{du}{6x^2} = \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\ln u = \frac{1}{6}\ln\left(1 + 2x^3\right)\bigg|_0^1 = \frac{1}{6}\ln(3 - \ln 1) = \frac{1}{6}\ln 3$$

Ex-26-8 Fig-26-8 shows part of the curve $y = x\cos 2x$, together with a point P at which the curve crosses the x-axis

Fig-26-8
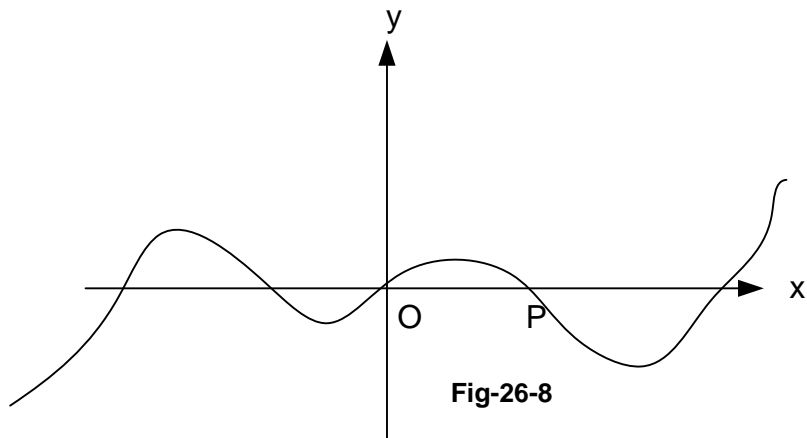
(i) Find the exact coordinates of P

$y = x\cos 2x$

$at\ P\ y = 0\ \Rightarrow x\cos 2x = 0\ \Rightarrow x = 0\ \ and\ \ \cos 2x = 0\ \Rightarrow 2x = \dfrac{\pi}{2}\ \Rightarrow x = \dfrac{\pi}{4}$

$\Rightarrow P\left(\dfrac{\pi}{4}, 0\right)$

(ii) Show algebraically that $x\cos 2x$ is an odd function, and interpret this result graphically.

$y = f(x) = x\cos 2x$

$f(-x) = -x\cos(-2x) - x\cos 2x = -f(x)\ \ \Rightarrow odd$

Fig-26-8

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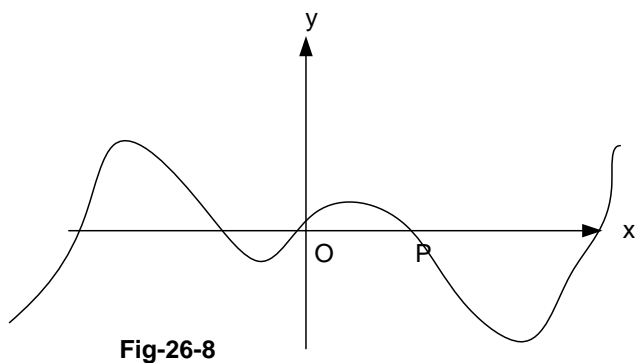
(iii) Find $\dfrac{dy}{dx}$

$y = x\cos 2x$

$\dfrac{dy}{dx} = x(-2\sin 2x) + \cos 2x = -2x\sin 2x + \cos 2x$

(iv) Show that turning points occur on the curve for values of x, which

satisfy the equation curve $x\tan 2x = \dfrac{1}{2}$

$\dfrac{dy}{dx} = -2x\sin 2x + \cos 2x = 0 \Rightarrow 2x\sin 2x = \cos 2x \Rightarrow \dfrac{2x\sin 2x}{\cos 2x} = 2x\tan 2x = 1 \Rightarrow x\tan 2x = \dfrac{1}{2}$

(v) Find the gradient of the curve at the origin.

$\dfrac{dy}{dx} = -2x\sin 2x + \cos 2x\Big|_{x=0} = 0 + 1 = 1$

Show that the second derivative of $x\cos 2x$ is zero when $x = 0$.

$y = x\cos 2x$

$\dfrac{dy}{dx} = -2x\sin 2x + \cos 2x$

$\dfrac{d^2 y}{dx^2} = -2(2x\cos 2x + \sin 2x) - 2\sin 2x\Big|_{x=0} = 0$

(vi) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} x\cos 2x\,dx$, giving your answer in terms of π. Interpret this

result graphically.

$\displaystyle\int_0^{\frac{\pi}{4}} x\cos 2x\,dx$

$u = x \qquad and \qquad dv = \cos 2x\,dx$

$\dfrac{du}{dx} = 1 \qquad \Rightarrow du = dx \qquad and \quad v = \dfrac{1}{2}\sin 2x$

$$\int_0^{\frac{\pi}{4}} x\cos 2x\,dx = \int_0^{\frac{\pi}{4}} u\,dv = uv - \int v\,du = \frac{x}{2}\sin 2x - \int \frac{1}{2}\sin 2x\,dx = \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\Big|_0^{\frac{\pi}{4}}$$

$$\int_0^{\frac{\pi}{4}} x\cos 2x\,dx = \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x\Big|_0^{\frac{\pi}{4}} = \frac{\pi}{8}\sin\frac{2\pi}{4} + \frac{1}{4}\cos\frac{2\pi}{4} - \left(\frac{1}{4}\right) = \frac{\pi}{8} + 0 - \frac{1}{4} = \frac{\pi-2}{8}$$

PAPER-27

SECTION A (C3-11-1-08)

Ex-27-1: Differentiate $\sqrt[3]{1+6x^2}$.

$y = \sqrt[3]{1+6x^2}$

$y = \left(1+6x^2\right)^{\frac{1}{3}}$

$\dfrac{dy}{dx} = \dfrac{1}{3}\left(1+6x^2\right)^{-\frac{2}{3}}(12x) = \dfrac{12x}{3}\left(1+6x^2\right)^{-\frac{2}{3}} = 4x\left(1+6x^2\right)^{-\frac{2}{3}}$

Ex-27-2: The function $f(x)$ and $g(x)$ are defined for all real numbers x by

$f(x) = x^2$ and $g(x) = x - 2$

(i) Find the composite functions $fg(x)$ and $gf(x)$.

$fg(x) = f(x-2) = (x-2)^2$

$gf(x) = g(x^2) = x^2 - 2$

(ii) (ii) Sketch the curves $y = f(x)$, $y = fg(x)$ and $y = gf(x)$ indicating clearly which is which.
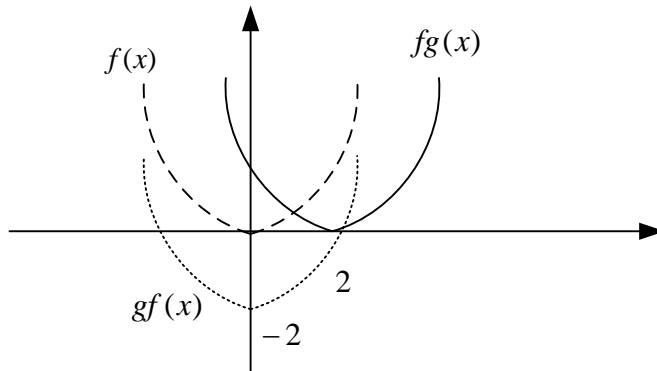
Ex-27-3: The profit £P made by a company in its nth year is modelled by the exponential function

$$P = Ae^{bn}$$

In the first year (when n=1), the profit was £10000. In the second year, the profit was £16000.

(i) Show that $e^b = 1.6$ and find b and A.

$$P = Ae^{bn}$$

$$10000 = Ae^b \qquad ...(1)$$

$$16000 = Ae^{2b} \qquad ...(2)$$

Divide Eq.2 by Eq.1

$$\frac{Ae^{2b}}{Ae^b} = e^b = \frac{16000}{10000} = 1.6 \qquad ...(3)$$

$$e^b = 1.6 \qquad \Rightarrow b \ln e = b = \ln 1.6 = 0.47$$

$$10000 = Ae^b = A = \frac{10000}{e^b} = \frac{10000}{e^{\ln 1.6}} = \frac{10000}{1.6} = 6250$$

(ii) What does this model predict the profit to be in the 2oth year?

$$P = Ae^{bn} = 6250\, e^{20(0.47)} = £75550000$$

Ex-27-4: When the gas in the balloon is kept at a constant temperature, the pressure P in atmospheres and the volume $V\, m^3$ are related by the equation, $P = \dfrac{k}{V}$.

Where k is a constant. [This is known as Boyle's Law]

When the volume is $100 \, m^3$, the pressure is 5 atmospheres, and the volume is increasing at a rate of $10 \, m^3$ per second.

(i) Show that $k = 500$.

$$P = \frac{k}{V}$$

$$5 = \frac{k}{100} \implies k = 500$$

(ii) Find $\dfrac{dP}{dV}$ in terms of V.

$$P = \frac{k}{V} = kV^{-1}$$

$$\frac{dP}{dV} = -kV^{-2} = -\frac{k}{V^2} = -\frac{500}{V^2}$$

(iii) Find the rate at which the pressure is decreasing when $V = 100$

$$\frac{dP}{dt} = \frac{dP}{dV} \times \frac{dV}{dt}$$

$$\frac{dP}{dV}\bigg|_{V = 100} = -\frac{500}{V^2} = -\frac{500}{10000} = -0.05$$

$$\frac{dP}{dt} = \frac{dP}{dV} \times \frac{dV}{dt} = (-0.05)(10) = -0.5 \frac{Atm}{s}$$

Ex-27-5: (i) Verify the following statement:

'$2^p - 1$ is a prime number for all prime numbers p less than 11'

$p = 2 \implies 2^2 - 1 = 4 - 1 = 3 \implies prime \ number$

$p = 3 \implies 2^3 - 1 = 8 - 1 = 7 \implies prime \ number$

$p = 5 \implies 2^5 - 1 = 32 - 1 = 31 \implies prime \ number$

$p = 7 \implies 2^7 - 1 = 128 - 1 = 127 \implies prime \ number$

(iii) Calculate 23×89, and hence disprove this statement: '$2^p - 1$ is a prime number for all prime numbers p'

Ex-27-6: Fig-27-6 shows the curve $e^{2y} = x^2 + y$.

Fig-27-6
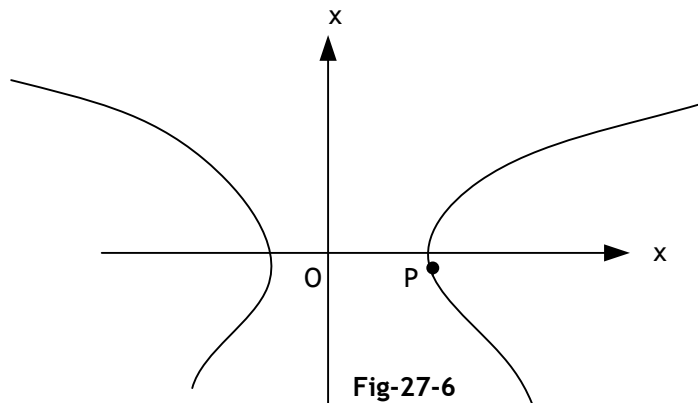
(i)   Show that $\dfrac{dy}{dx} = \dfrac{2x}{2e^{2y}-1}$.

$$e^{2y} = x^2 + y$$

$$e^{2y}\left(2\dfrac{dy}{dx}\right) = 2x + \dfrac{dy}{dx} \Rightarrow \left(2e^{2y}-1\right)\dfrac{dy}{dx} = 2x \quad \Rightarrow \dfrac{dy}{dx} = \dfrac{2x}{2e^{2y}-1}$$

(ii)   Hence find to 3 significant figures the coordinates of the point P, shown in Fig.6, where the curve has infinite gradient.

$$\Rightarrow \dfrac{dy}{dx} = \dfrac{2x}{2e^{2y}-1} \to \infty \Rightarrow 2e^{2y} - 1 = 0 \quad \Rightarrow e^{2y} = 0.5 \quad \Rightarrow 2y = \ln(0.5)$$

$$\Rightarrow y = \dfrac{\ln(0.5)}{2} = -0.347\,(3\,s.f)$$

$$e^{2y} = x^2 + y$$

$$x^2 = e^{2y} - y = e^{2(-0.347)} - (-0.347) = 0.8467$$

$$x^2 = 0.8467 \quad \Rightarrow x = 0.920$$

SECTION B (P-27)

Ex-27-7: A curve is defined by the equation $y = 2x\ln(1+x)$.

(i)   Find $\dfrac{dy}{dx}$, and hence verify that the origin is a stationary point of the curve.

$$y = 2x\ln(1+x)$$

$$\frac{dy}{dx} = 2x\left(\frac{1}{1+x}\right) + 2\ln(1+x) = \frac{2x}{1+x} + 2\ln(1+x) = \frac{2x + 2(1+x)\ln(1+x)}{1+x}$$

$$\frac{dy}{dx} = \frac{2x + 2(1+x)\ln(1+x)}{1+x}\bigg|_{x=0} = \frac{0 + 2\ln 1}{1} = 0$$

(ii)   Find $\frac{d^2y}{dx^2}$, and use this to verify that the origin is a minimum point.

$$\frac{dy}{dx} = \frac{2x}{1+x} + 2\ln(1+x)$$

$$\frac{d^2y}{dx^2} = \frac{(1+x)(2) - 2x}{(1+x)^2} + \frac{2}{(1+x)}\bigg|_{x=0} = \frac{2}{1} + 2 = 4 \phi 0$$

Because $\frac{d^2y}{dx^2} = 4 \phi 0$, then point P(0,0) is a minimum.

Using the substitution $= 1 + x$, show that $\int \frac{x^2}{1+x} dx = \int \left(u - 2 + \frac{1}{u}\right) du$ hence evaluate

$\int_0^1 \frac{x^2}{1+x} dx$, giving your answer in an exact form.

$$\int \frac{x^2}{1+x} dx$$

$let \ \ u = 1 + x \ \ \Rightarrow du = dx$

And $and \ \ x = u - 1 \ \ \Rightarrow x^2 = (u-1)^2$

$$\int \frac{x^2}{1+x} dx = \int \frac{(u-1)^2}{u} du = \int \frac{u^2 - 2u + 1}{u} du = \int \left(u - 2 + \frac{1}{u}\right) du$$

$u = 1 + x$

$u = 1 + x\bigg|_{x=0} = 1$

$u = 1 + x\bigg|_{x=1} = 2$

$$\int_0^1 \frac{x^2}{1+x}dx = \int_1^2 \left(u - 2 + \frac{1}{u}\right)du = \frac{u^2}{2} - 2u + \ln u \Big|_1^2 = \frac{4}{2} - 4 + \ln 2 - \left(\frac{1}{2} - 2 + \ln 1\right)$$

$$\int_0^1 \frac{x^2}{1+x}dx = \ln 2 - 2 - \frac{1}{2} + 2 = \ln 2 - \frac{1}{2}$$

Using integration by parts and your answer to part (iii),

evaluate $\int_0^1 2x\ln(1+x)dx.$

$$\int_0^1 2x\ln(1+x)dx$$

$let \quad u = \ln(1+x) \quad and \quad dv = 2xdx$

$$\frac{du}{dx} = \frac{1}{1+x} \quad and \quad v = \frac{2x^2}{2} = x^2$$

$$\int_0^1 2x\ln(1+x)dx = \int udv = \left[uv - \int vdu\right] = x^2\ln(1+x) - \int \frac{x^2}{1+x}dx = x^2\ln(1+x)\Big|_0^1 + \ln 2 - \frac{1}{2}$$

$$\int_0^1 2x\ln(1+x)dx = x^2\ln(1+x)\Big|_0^1 + \ln 2 - \frac{1}{2} = \ln 2 + \ln 2 - \frac{1}{2} = 2\ln 2 - \frac{1}{2}$$

Ex-27-8: Fig-27-8 shows the curve $y = f(x)$, where $f(x) = 1 + \sin 2x$ for

$-\frac{1}{4}\pi \le x \le \frac{1}{4}\pi$

Fig-27-8
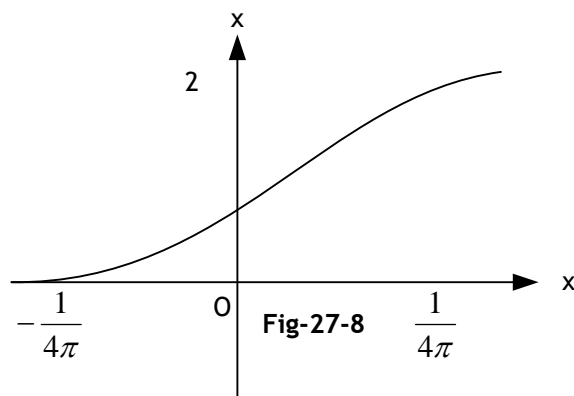
(i) State a sequence of two transformations that would map part of the curve $y = \sin x$ onto the curve $y = f(x)$

Ans: The steps are as follows:

(a) $f(x) = f(x) - 1$

(b) $f(x) = f\left(\dfrac{x}{2}\right)$

(ii) Find the area of the region enclosed by the curve $y = f(x)$, the x-axis and the line $x = \dfrac{1}{4}\pi$.

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \sin 2x)\,dx = x - \frac{1}{2}\cos 2x \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{1}{2}\cos\frac{2\pi}{4} - \left(-\frac{\pi}{4} - \frac{1}{2}\cos\frac{-2\pi}{4}\right)$$

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 + \sin 2x)\,dx = x - \frac{1}{2}\cos 2x \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{\pi}{4} - \frac{1}{2}\cos\frac{2\pi}{4} + \frac{\pi}{4} + \frac{1}{2}\cos\frac{2\pi}{4} = \frac{\pi}{4} - 0 + \frac{\pi}{4} + 0 = \frac{\pi}{2}$$

(iii) Find the gradient of the curve $y = f(x)$ at the point (0, 1).

$$y = f(x) = 1 + \sin 2x$$

$$gradient = \frac{dy}{dx} = 2\cos 2x \Big|_{x=0} = 2$$

Hence write down the gradient of the curve $y = f^{-1}(x)$ at the point (1,0).

$$y = 1 + \sin 2x$$

$$\sin 2x = y - 1$$

$$2x = \sin^{-1}(y - 1)$$

$$x = \frac{1}{2}\sin^{-1}(y - 1)$$

$$f^{-1}(y) = \sin^{-1}(y - 1)$$

$$f^{-1}(x) = \sin^{-1}(x - 1)$$

$$y = f(x) = \sin^{-1}(x - 1) \qquad \Rightarrow x - 1 = \sin y$$

$$x - 1 = \sin y$$

$$1 = \cos y \frac{dy}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}\bigg|_{y=0} = 1$$

(iv)   Sketch the domain of $f^{-1}(x)$. Add a sketch of $y = f^{-1}(x)$ to a copy of Fig.8.

(v)   Find an expression for $f^{-1}(x)$.

PAPER-28

SECTION A (C3-2-6-08)

Ex-28-1:   solve the inequality $|1 - 2x| \leq 3$.

$$|1 - 2x| \leq 3$$

$$-3 \leq (1 - 2x) \leq 3$$

(a) $(1 - 2x) \leq 3 \quad \Rightarrow -2x \leq 3 - 1 \quad \Rightarrow -2x \leq 2 \quad \Rightarrow x \geq -1$

(b) $(1 - 2x) \geq -3 \quad \Rightarrow -2x \geq -3 - 1 \quad \Rightarrow 2x \leq 3 + 1 \quad \Rightarrow 2x \leq 4 \Rightarrow x \leq 2$

Ans: $-1 \leq x \leq 2$

Ex-28-2:   Find $\int 3xe^{3x} dx$

$$\int 3xe^{3x} dx = \int u dv = uv - \int v du \quad \text{int } egration \quad by \quad part.$$

$$Let \quad u = 3x \quad and \quad dv = e^{3x} dx$$

$$du = 3dx \quad and \quad v = \int e^{3x} dx = \frac{1}{3} e^{3x}$$

$$\int xe^{3x} dx = \int u dv = uv - \int v du = \frac{3x}{3} e^{3x} - \int \frac{3}{3} e^{3x} dx = xe^{3x} - \frac{1}{3} e^{3x} + C.$$

Ex-28-3:   (i)   State the algebraic condition for the function $f(x)$ to be an

even function. What geometrical property does the graph of an even function have?

(a)   If $f(-x) = f(x)$ Then $f(x)$ is   Even

(b)   If $f(-x) = -f(x)$ Then $f(x)$ is   Odd

(c)   If $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$ Then   is   neither

(ii)   State whether the following functions are odd, even or neither.

(A) $f(x) = 3 - x^2$.

$$f(-x) = 3 - (-x)^2 = 3 - x^2 = f(x) \quad \Rightarrow even$$

(B) $g(x) = \cos x - \sin x$.

$$g(-x) = \cos(-x) - \sin(-x) = \cos x + \sin x \neq f(x) \qquad \Rightarrow neither$$

(C) $h(x) = \dfrac{1}{x^3 + x}$.

$$h(-x) = \frac{1}{(-x)^3 + (-x)} = \frac{1}{-x^3 - x} = -\frac{1}{x^3 + x} = -h(x) \quad \Rightarrow odd$$

Ex-28-4:   Show that $\displaystyle\int_1^4 \frac{x}{3x^2 + 2}dx = \frac{1}{6}\ln 10$

$$Let \quad u = 3x^2 + 2 \quad \Rightarrow \frac{du}{dx} = 6x \Rightarrow dx = \frac{du}{6x}$$

$$\int_1^4 \frac{x}{x^2 + 2}dx = \int \frac{x}{u}\frac{du}{6x} = \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\ln u = \frac{1}{6}\ln(3x^2 + 2)\Big|_1^4 = \frac{1}{6}\ln(48 + 2) - \frac{1}{6}\ln(5) = \frac{1}{6}\ln\left(\frac{50}{5}\right) = \frac{1}{6}\ln 10$$

Ex-28-5:   Show that the curve $y = x\ln x$ has a stationary point when $x = \dfrac{1}{e}$

$$y = x\ln x$$

$$\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x = 0 \quad \Rightarrow 1 + \ln x = 0 \quad \Rightarrow \ln x = -1 \Rightarrow x = e^{-1} = \frac{1}{e}$$

Ex-28-6:   In a chemical reaction, the mass m grams of a chemical after t minutes is modelled by the equation $m = 40 - 30e^{-0.2t}$.

(i) Find the initial mass of the chemical. What is the mass of the chemical in the long term?

$$m = 40 - 30e^{-0.2t}\Big|_{t=0} = 40 - 30 = 10mg$$

$$m = 40 - 30e^{-0.2t}\Big|_{t \to \infty} = 40 - 30e^{-\infty} = 40 - \frac{30}{e^{\infty}} = 40 - 0 = 40mg$$

(ii) Find the time when the mass is 30 grams.

$$30 = 40 - 30e^{-0.2t} \Rightarrow -30e^{-0.2t} = 30 - 40 = -10 \Rightarrow e^{-0.2t} = \frac{-10}{-30} = \frac{1}{3}$$

$$\Rightarrow e^{-0.2t} = \frac{1}{3} \Rightarrow -0.2t = \ln 1 - \ln 3 = 0 - \ln 3 \Rightarrow t = \frac{-\ln 3}{-0.2} = 5\ln 3$$

(iii) Sketch the graph of m against t.

$$m = 40 - 30e^{-0.2t}$$
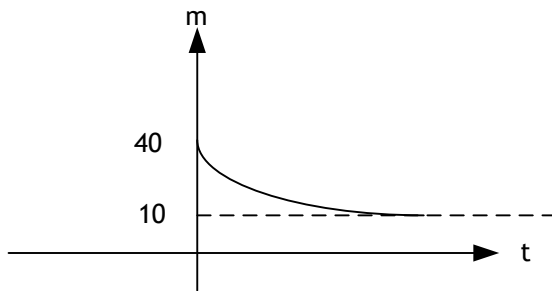
Ex-28-7: Given that $x^2 - 2xy + 3y^2 = 10$, find $\dfrac{dy}{dx}$ in terms of x and y.

$$x^2 - 2xy + 3y^2 = 10$$

$$2x - 2x\frac{dy}{dx} - 2y + 6y\frac{dy}{dx} = 0 \quad \Rightarrow (6y - 2x)\frac{dy}{dx} = (2y - 2x) \Rightarrow \frac{dy}{dx} = \frac{(2y - 2x)}{6y - 2x}$$

Ex-28-8: Fig-28-8 shows the curve $y = f(x)$, where $f(x) = \dfrac{1}{1 + \cos x}$, for $0 \le x \le \dfrac{1}{2}\pi$. P is the point on this curve with x-coordinate $\dfrac{1}{6}\pi$.

Fig-28-8
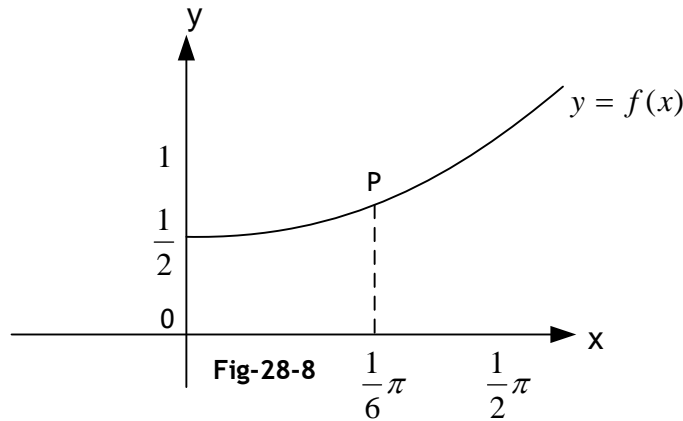
(i) Find the y-coordinate of P.

$$f(x) = \frac{1}{1 + \cos x}$$

$$y\bigg|_{x = \frac{\pi}{6}} = \frac{1}{1 + \cos x}\bigg|_{x = \frac{\pi}{6}} = \frac{1}{1 + \cos\left(\frac{\pi}{6}\right)} = \frac{1}{1 + 0.866} = \frac{1}{1.866} = 0.536$$

(ii) Find $f'(x)$. Hence find the gradient of the curve at point P.

$$f(x) = \frac{1}{1 + \cos x}$$

$$f(x) = \frac{1}{1 + \cos x} = (1 + \cos x)^{-1}$$

$$f'(x) = \frac{df}{dx} = -1(1 + \cos x)^{-2}(-\sin x) = \frac{\sin x}{(1 + \cos x)^2}$$

$$f'(x) = \frac{df}{dx} = \frac{\sin x}{(1 + \cos x)^2}\bigg|_{x = \frac{\pi}{6}} = \frac{\sin\left(\frac{\pi}{6}\right)}{\left(1 + \cos\left(\frac{\pi}{6}\right)\right)^2} = \frac{0.5}{(1 + 0.866)^2} = \frac{0.5}{3.482} = 0.1436$$

(iii) Show that the derivative of $\dfrac{\sin x}{1 + \cos x}$ is $\dfrac{1}{1 + \cos x}$. Hence find the exact area of the region enclosed by the curve $y = f(x)$, the x-axis, the y-axis and the line $x = \frac{1}{3}\pi$.

$$y = \frac{\sin x}{1 + \cos x} = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v \times \dfrac{du}{dx} - u \times \dfrac{dv}{dx}}{v^2} = \frac{(1+\cos x)(\cos x) - \sin x(-\sin x)}{(1+\cos x)^2} = \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$\frac{dy}{dx} = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

(iv)   Show that $f^{-1}(x) = \arccos\left(\dfrac{1}{x} - 1\right)$. State the domain of this inverse

function, and add a sketch of $y = f^{-1}(x)$ to a copy of Fig.8.

$$f(x) = \frac{1}{1+\cos x}$$

$$y = \frac{1}{1+\cos x} \quad \Rightarrow y + y\cos x = 1 \quad \Rightarrow y\cos x = 1 - y \Rightarrow \cos x = \frac{1-y}{y}$$

$$\cos x = \frac{1-y}{y} \quad \Rightarrow x = \arccos\left(\frac{1-y}{y}\right)$$

$$\Rightarrow f^{-1}(y) = \arccos\left(\frac{1-y}{y}\right)$$

$$\Rightarrow f^{-1}(x) = \arccos\left(\frac{1-x}{x}\right)$$

Ex-28-9:   The function $f(x)$ is defined by $f(x) = \sqrt{4 - x^2}$ for $-2 \le x \le 2$.
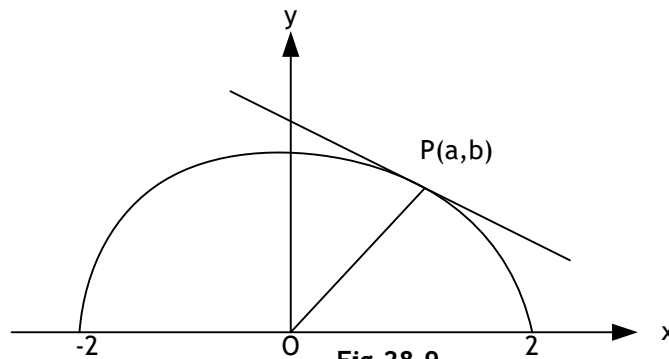
Fig-28-9

(i)   Show that the curve $y = \sqrt{4 - x^2}$ is a semicircle of radius 2, and
explain why it is not the whole of this circle.

$$y = \sqrt{4 - x^2}$$

$$4 - x^2 \ge 0 \quad \Rightarrow -x^2 \ge -4 \quad \Rightarrow x^2 \le 4 \quad -2 \le x \le 2$$

For this values of $-2 \le x \le 2$ the value of $y = \sqrt{4-x^2}$ is always positive

$$y = \sqrt{4-x^2}$$

$$y^2 = 4-x^2 \qquad \Rightarrow x^2 + y^2 = 4 = r^2 \qquad \Rightarrow r = 2$$

Fig-28-9 shows a point P(a, b) on the semicircle. The tangent at P is shown.

(ii)   (A)   Use the gradient of OP to find the gradient of the tangent at P in terms of a and b.

$$y = \sqrt{4-x^2} = \left(4-x^2\right)^{\frac{1}{2}} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(4-x^2\right)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{4-x^2}}\bigg|_{x=a} = -\frac{a}{\sqrt{4-a^2}}$$

$$-\frac{a}{\sqrt{4-a^2}} = -\frac{a}{b}, \; because \; y^2\bigg|_{y=b} = b^2 = 4-x^2\bigg|_{at \; P(x=a)} = 4-a^2$$

(B)   Differentiate $\sqrt{4-x^2}$ and deduce the value of $f'(a)$.

$$y = \sqrt{4-x^2} = \left(4-x^2\right)^{\frac{1}{2}} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(4-x^2\right)^{-\frac{1}{2}}(-2x)$$

$$f'(x) \Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(4-x^2\right)^{-\frac{1}{2}}(-2x) = -\frac{x}{\sqrt{4-x^2}} \quad \Rightarrow f'(a) = -\frac{a}{\sqrt{4-a^2}}$$

(C)   Show that your answers to parts (A) and (B) are equivalent.

The function $g(x)$ is defined by $g(x) = 3f(x-2), \quad for \; 0 \le x \le 4.$

(iii)   Describe a sequence of two transformations that would map the curve $y = f(x)$ onto the curve $y = g(x)$. Hence sketch the curve $y = g(x)$.

(iv)   Show that if $y = g(x)$ *then* $9x^2 + y^2 = 36x.$.

$$g(x) = 3f(x-2) = 3\left(\sqrt{4-(x-2)^2}\right) = 3\left(\sqrt{4-x^2+4x-4}\right) = 3\sqrt{4x-x^2}$$

$$g(x) = f(x) \quad \Rightarrow 3\sqrt{4x-x^2} = \sqrt{4-x^2} \Rightarrow 9\left(4x-x^2\right) = 4-x^2$$

$$36x - 9x^2 = 4-x^2 = y^2 \qquad \Rightarrow 9x^2 + y^2 = 63x$$

PAPER-29

SECTION A (C3-15-1-09)

Ex-29-1: Solve the equation $|2-x| < 3$.

$$-3 < (2-x) < 3$$

(a) $(2-x) < 3 \Rightarrow 2-x < 3 \qquad \Rightarrow -x < 3-2 = 1 \qquad \Rightarrow x > -1$

(b) $(2-x) > -3 \Rightarrow -x > -3-5 \Rightarrow x < 3+2 \Rightarrow x < 5$

Ans: $-1 < x < 5$

Ex-29-2: (i) Differentiate $x\cos 2x$ with respect to x

(ii) Integrate $x\cos 2x$ with respect to x

(a) If if $y = u \times v$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

$y = x\cos 2x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x(-2\sin 2x) + \cos 2x = -2x\sin 2x + \cos 2x$$

(b)

If $\displaystyle\int x\cos 2x dx$

$\displaystyle\int x\cos 2x dx = \int u dv = uv - \int v du \quad int\,egration \quad by \quad part.$

$Let \quad u = x \quad and \quad dv = \cos 2x dx$

$du = dx \quad and \quad v = \displaystyle\int \cos 2x dx = \frac{1}{2}\sin 2x$

$$\int x\cos 2x dx = \int u dv = uv - \int v du = \frac{x}{2}\sin 2x - \int \frac{1}{2}\sin 2x dx = \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C.$$

Ex-29-3: Given that $f(x) = \dfrac{1}{2}\ln(x-1)$ and $g(x) = 1 + e^{2x}$, show that $g(x)$ is the inverse of $f(x)$.

Note: $\log x = y \qquad \Rightarrow x = 10^y$ and $\ln x = y \qquad \Rightarrow x = e^y$

$\qquad \log(x-1) = y \qquad \Rightarrow x - 1 = 10^y \qquad \Rightarrow x = 1 + 10^y$

$\qquad \ln(x-1) = y \qquad \Rightarrow x - 1 = e^y \qquad \Rightarrow x = 1 + e^y$

$f(x) = \dfrac{1}{2}\ln(x-1)$

$y = \dfrac{1}{2}\ln(x-1) \qquad \Rightarrow \ln(x-1) = 2y \qquad \Rightarrow x - 1 = e^{2y} \qquad \Rightarrow x = 1 + e^{2y}$

$\Rightarrow f^{-1}(y) = 1 + e^{2y} \qquad\qquad \Rightarrow f^{-1}(x) = 1 + e^{2x} = g(x) \qquad OK$

Ex-29-4:   Find the exact value of $\displaystyle\int_0^2 \sqrt{1+4x}\,dx$, showing your working.

$Let \quad u = 1 + 4x \qquad \Rightarrow \dfrac{du}{dx} = 4 \qquad \Rightarrow du = 4dx \Rightarrow dx = \dfrac{du}{4}$

$$\int_0^2 \sqrt{1+4x}\,dx = \int u^{\frac{1}{2}}\left(\frac{du}{4}\right) = \frac{1}{4}\int u^{\frac{1}{2}}\,du = \frac{1}{4\left(\frac{3}{2}\right)}u^{\frac{3}{2}} = \frac{2}{12}u^{\frac{3}{2}} = \left.\frac{(1+4x)^{\frac{3}{2}}}{6}\right|_0^2 = \frac{1}{6}\left[(1+8)^{\frac{3}{2}} - (1+0)^{\frac{3}{2}}\right]$$

$$\int_0^2 \sqrt{1+4x}\,dx = \frac{1}{6}\left[(9)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right] = \frac{1}{6}(27-1) = \frac{26}{6} = \frac{13}{3}$$

Ex-29-5:   (i)   State the period of a the function $f(x) = 1 + \cos 2x$, where x is in degrees.

$y = 1 + \cos x$

$y = \cos x$
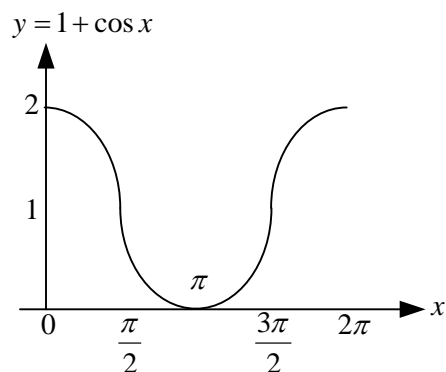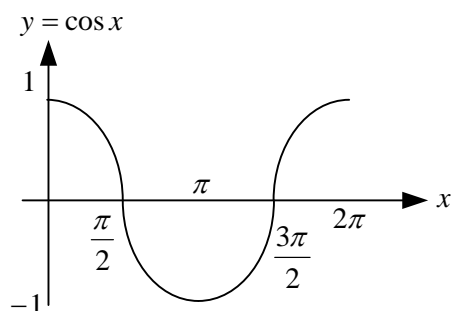
(ii)　　State a sequence of two geometrical transformations which maps the curve $y = \cos x$, onto the curve $y = f(x)$.

(iii)　Sketch the graph of $y = f(x)$ for $-180^0 < x < 180^0$

$$y = 1 + \cos 2x$$
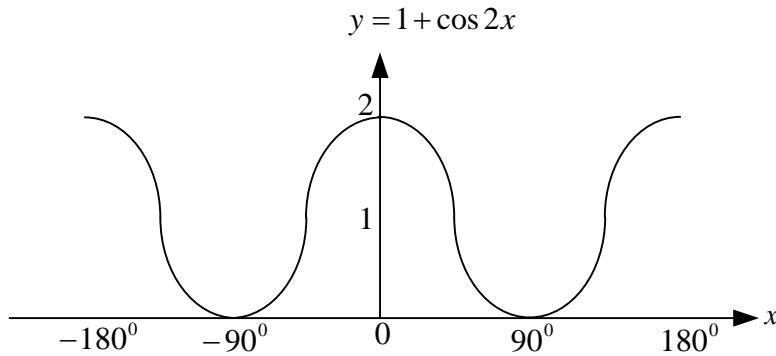
Ex-29-6:　　(i)　　Disprove the following statement

$'$ If $p > q$, then $\dfrac{1}{p} < \dfrac{1}{q}$ $'$

Let $\quad p = 1, \quad q = -2$

$\Rightarrow p > q$

But $\quad \dfrac{1}{p} = 1 > \dfrac{1}{q} = -\dfrac{1}{2}$

(ii)　　State a condition on p and q so that the statement is true.

Both p and q should be positive or both negative.

Ex-29-7:　　The variables x and y satisfy the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5$.

(i)　　Show that $\dfrac{dy}{dx} = -\left(\dfrac{y}{x}\right)^{\frac{1}{3}}$. Both x and y are functions of t.

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5 \quad \Rightarrow \frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0 \quad \Rightarrow \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = -\frac{2}{3}x^{-\frac{1}{3}} \quad \Rightarrow \frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} = -\sqrt[3]{\frac{y}{x}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

(ii) Find the value of $\dfrac{dy}{dt}$ when $x=1,\ \ y=8\ \ and\ \ \dfrac{dx}{dt}=6$.

$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}=-\left(\frac{y}{x}\right)^{\frac{1}{3}}\Bigg|_{x=1\ \ and\ \ y=8}\times 6=-\left(\frac{8}{1}\right)^{\frac{1}{3}}\times 6=-12$$

Ex-29-8: Fig-29-8 shows the curve $y=x^{2}-\dfrac{1}{8}\ln x$. P is the point on this curve

with x-coordinate 1, and R is the point $\left(0,-\dfrac{7}{8}\right)$.

Fig-29-8
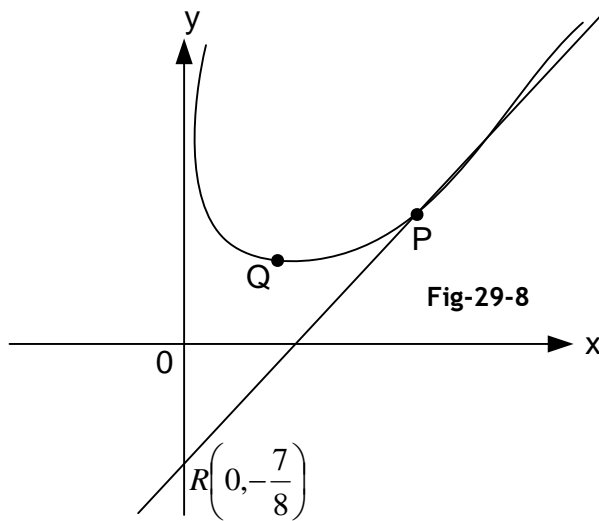
(v) Find the gradient of PR.

Gradient $m=\dfrac{y_{2}-y_{1}}{x_{2}-x_{1}}$

Find $y\Big|_{x=1}=x^{2}-\dfrac{1}{8}\ln x\Big|_{x=1}=1-\dfrac{1}{8}\times 0=1\quad\Rightarrow P(1,1)$.

$$\Rightarrow m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}=\frac{1-\left(-\dfrac{7}{8}\right)}{1-0}=1\frac{7}{8}$$

(vi) Find $\dfrac{dy}{dx}$. Hence show that PR is a tangent to the curve.

$$y=x^{2}-\frac{1}{8}\ln x$$

$$\frac{dy}{dx} = 2x - \frac{1}{8}\left(\frac{1}{x}\right) = 2x - \frac{1}{8x}\bigg|_{x=1} = 2 - \frac{1}{8} = 1\frac{7}{8}$$

(vii)   Find the exact coordinates of the turning point Q.

$$\frac{dy}{dx} = 2x - \frac{1}{8}\left(\frac{1}{x}\right) = 2x - \frac{1}{8x} = 0 \;\Rightarrow\; \frac{16x^2 - 1}{8x} = 0 \;\Rightarrow\; 16x^2 = 1 \quad \Rightarrow x = \frac{1}{4}$$

$$y = x^2 - \frac{1}{8}\ln x\bigg|_{x=\frac{1}{4}} = \left(\frac{1}{4}\right)^2 - \frac{1}{8}\ln\left(\frac{1}{4}\right) = \frac{1}{16} - \frac{1}{8}\ln(2)^{-2} = \frac{1}{16} + \frac{1}{4}\ln 2$$

(viii)   Differentiate $x\ln x - x$. Henc, or otherwise, show that the area of the region enclosed by the curve $y = x^2 - \frac{1}{8}\ln x$, the x-axis and the lines $x = 1$ and $x = 2$ is $\frac{59}{24} - \frac{1}{4}\ln 2$.

$x\ln x - x$

$$y = x\ln x - x \quad \Rightarrow \frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x - 1 = 1 + \ln x - 1 = \ln x$$

$$\int_1^2 \left(x^2 - \frac{1}{8}\ln x\right)dx = \frac{x^3}{3}\bigg|_1^2 - \frac{1}{8}(x\ln x - x)\bigg|_1^2 = \frac{8}{3} - \frac{1}{3} - \frac{1}{8}(2\ln 2 - 2 - \ln 1 + 1) = \frac{7}{3} - \frac{1}{4}\ln 2 + \frac{1}{8}$$

Ex-29-9   Fig-29-9 shows the curve $y = f(x)$, where $f(x) = \dfrac{1}{\sqrt{2x - x^2}}$. The curve has asymptotes x=0 and x=a.

Fig-29-9
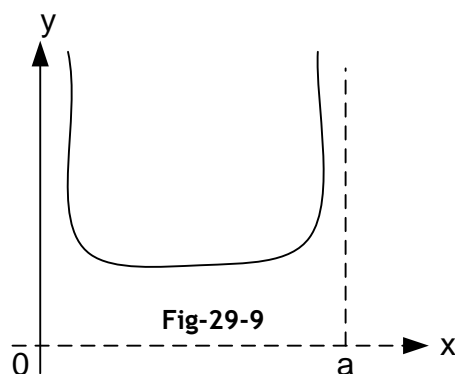
(i)   Find a. Hence write down the domain of the function.

$$f(x) = \frac{1}{\sqrt{2x - x^2}}$$

For asymptotes

$$f(x) \to \infty \qquad \sqrt{2x - x^2} = 0 \quad \Rightarrow 2x - x^2 = 0 \quad x(2 - x) = 0 \quad x = 2 = a$$

(ii)   Show that $\dfrac{dy}{dx} = \dfrac{x-1}{\left(2x - x^2\right)^{\frac{3}{2}}}$.

$$y = \frac{1}{\sqrt{2x - x^2}} = \left(2x - x^2\right)^{-\frac{1}{2}} \quad \Rightarrow \frac{dy}{dx} = -\frac{1}{2}\left(2x - x^2\right)^{-\frac{3}{2}}(2 - 2x) = -\frac{2 - 2x}{2}\left(2x - x^2\right)^{-\frac{3}{2}}$$

$$\Rightarrow \frac{dy}{dx} = -(1 - x)\left(2x - x^2\right)^{-\frac{3}{2}} = -\frac{1 - x}{\left(2x - x\right)^{\frac{3}{2}}} = \frac{x - 1}{\left(2x - x\right)^{\frac{3}{2}}}$$

The function $g(x)$ is defined by $g(x) = \dfrac{1}{\sqrt{1 - x^2}}$.

(iii)   (A)   Show algebraically that $g(x)$ is an even function.

$$g(x) = \frac{1}{\sqrt{1 - x^2}} \qquad g(-x) = \frac{1}{\sqrt{1 - (-x)^2}} = \frac{1}{\sqrt{1 - x^2}} = g(x)$$

$$\Rightarrow g(-x) = g(x) \qquad even$$

(B)   Show that $g(x - 1) = f(x)$.

$$g(x) = \frac{1}{\sqrt{1 - x^2}} \quad \Rightarrow g(x - 1) = \frac{1}{\sqrt{1 - (x - 1)^2}} = \frac{1}{\sqrt{1 - \left(x^2 - 2x + 1\right)}} = \frac{1}{\sqrt{2x - x^2}}$$

$$\Rightarrow g(x - 1) = \frac{1}{\sqrt{2x - x^2}} = f(x)$$

(C)   Hence prove that the curve y=f(x) is symmetrical, and state its line of symmetry.

PAPER-30

Ex-30-1:   Evaluate $\displaystyle\int_{0}^{\frac{\pi}{6}} \sin 3x\,dx$.

$$\int_0^{\frac{\pi}{6}} \sin 3x\, dx = -\left.\frac{\cos 3x}{3}\right|_0^{\frac{\pi}{6}} = -\frac{1}{3}\left[\cos\left(\frac{3\pi}{6}\right) - \cos(0)\right] = -\frac{1}{3}\left[\cos\left(\frac{\pi}{2}\right) - \cos(0)\right] = -\frac{1}{3}[0 - 1] = \frac{1}{3}$$

Ex-30-2: A radioactive substance decays exponentially, so that its mass M grams can be modelled by the equation $M = Ae^{-kt}$, where t is the time in years, and A and k are positive constants.

(i) An initial mass of 100 grams of the substance decays to 50 grams in 1500 years. Find A and k.

$M = Ae^{-kt}$

$$Initial\ mass = \left.M\right|_{t=0} = \left.Ae^{-kt}\right|_{t=0} = Ae^0 = A = 100$$

$$\left.M\right|_{t=1500} = 50 = \left.Ae^{-kt}\right|_{t=1500} = \left.100\,e^{-kt}\right|_{t=1500} = 100\,e^{-1500k} = 50$$

$$\Rightarrow 100\,e^{-1500k} = 50 \quad \Rightarrow e^{-1500k} = 0.5 \quad \Rightarrow -1500\,k \ln e = \ln(0.5)$$

$$\Rightarrow -1500\,k = \ln(0.5) \quad \Rightarrow k = \frac{\ln 0.5}{-1500} = \frac{-0.69315}{-1500} = 4.621 \times 10^{-4}\ Years$$

(ii) The substance becomes safe when 99% of its initial mass has decayed. Find how long it will take before the substance becomes safe.

$$1\%\ of\ nitial\ mass = \frac{1}{100}(100) = 1\,gram$$

$$M = Ae^{-kt} \quad \Rightarrow 1 = 100e^{-4.62\times10^{-4}t} \quad \Rightarrow e^{-4.62\times10^{-4}} = 0.01$$

$$\Rightarrow -4.621 \times 10^{-4}t = \ln 0.01 \quad \Rightarrow t = \frac{-4.605}{-4.621 \times 10^{-4}} = 9965\ Years$$

Ex-30-3: Sketch the curve $y = 2\arccos x \quad for \ -1 \le x \le 1$.

Ex-30-4:   Fig-30-4 shows a sketch of a graph of $y = 2|x-1|$. It meets the x- and y-axis at (a, 0) and (0, b) respectively.

Fig-30-4
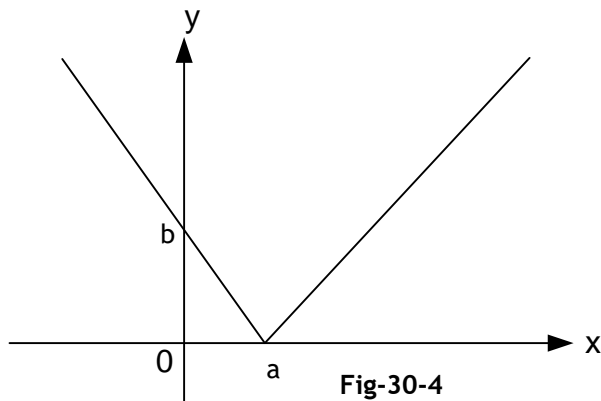
$y = 2|x-1|$

$y = \pm 2(x-1)$

(*i*) $y = 2(x-1) = 2x - 2$ $\Rightarrow y = 0 \ at \ x = b \Rightarrow 0 = 2a - 2$ $\Rightarrow a = 1$

(*ii*) $y = -2(x-1) = -2x + 2$ $\Rightarrow y = b \ at \ x = 0 \Rightarrow b = 2$

Find the values of **a** and **b**.

Ex-30-5:   The equation of a curve is given by $e^{2y} = 1 + \sin x$. Henc,

(i)   By differentiating implicitly, find $\dfrac{dy}{dx}$ in terms of x and y.

$e^{2y} = 1 + \sin x$

$e^{2y}\left(2\dfrac{dy}{dx}\right) = \cos x$

$2\dfrac{dy}{dx} = \dfrac{\cos x}{e^{2y}}$

$\dfrac{dy}{dx} = \dfrac{\cos x}{2e^{2y}}$

(ii)   Find an expression for y in terms of x, and differentiate it to verify the result in part (i).

$e^{2y} = 1 + \sin x$

$$2y \ln e = 2y = \ln(1 + \sin x)$$

$$y = \frac{1}{2}\ln(1 + \sin x)$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{\cos x}{1+\sin x}\right) = \frac{1}{2}\left(\frac{\cos x}{e^{2y}}\right)$$

Ex-30-6: Given that $f(x) = \dfrac{x+1}{x-1}$, show that ff(x)=x. Hence write down the inverse function $f^{-1}(x)$. What can you deduce about symmetry of the curve $y = f(x)$?.

$$f(x) = \frac{x+1}{x-1}$$

$$ff(x) = \frac{x+1}{x-1} = \frac{\dfrac{x+1}{x-1}+1}{\dfrac{x+1}{x-1}-1} = \frac{\dfrac{x+1+x-1}{x-1}}{\dfrac{x+1-x+1}{x-1}} = \frac{2x}{2} = x$$

$$f(x) = \frac{x+1}{x-1}$$

$$y = \frac{x+1}{x-1} \quad \Rightarrow y(x-1) = x+1 \quad \Rightarrow yx - y = x+1 \quad \Rightarrow yx - x = 1 + y \quad \Rightarrow x(y-1) = y+1$$

$$\Rightarrow x = \frac{y+1}{y-1} \quad \Rightarrow f^{-1}(y) = \frac{y+1}{y-1} \quad \Rightarrow f^{-1}(x) = \frac{x+1}{x-1} = f(x)$$

Ex-30-7: (i) Show that

(A) $(x-y)(x^2 + xy + y^2) = x^3 - y^3$,

$$(x-y)(x^2 + xy + y^2) = x^3 + x^2 y + xy^2 - yx^2 - xy^2 - y^3 = x^3 - y^3$$

(B) $\left(x + \dfrac{1}{2}y\right)^2 + \dfrac{3}{4}y^2 = x^2 + xy + y^2$,

$$\left(x + \frac{1}{2}y\right)^2 + \frac{3}{4}y^2 = x^2 + xy + \frac{1}{4}y^2 + \frac{3}{4}y^2 = x^2 + xy + y^2$$

(ii) Hence prove that, for all real numbers x and y, *if $x > y$ then $x^3 > y^3$*

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2) = (x-y)\left[\left(x + \frac{1}{2}y\right)^2 + \frac{3}{4}y^2\right]$$

$$\left(x + \frac{1}{2}y\right)^2 + \frac{3}{4}y^2 \geq 0$$

$$if \quad x > y \quad then \quad x - y > 0$$

$$x^3 - y^3 = (x-y)\left[\left(x + \frac{1}{2}y\right)^2 + \frac{3}{4}y^2\right] \geq 0$$

$$x^3 - y^3 \geq 0 \qquad \Rightarrow x^3 \geq y^3$$

Ex-30-8: Fig-30-8 shows the line $y = x$ and parts of the curves $y = f(x)$ and $y = g(x)$, where $f(x) = e^{x-1}$, $g(x) = 1 + \ln x$. The curves intersect the axes at the points A and B, as shown. The curves and the line $y = x$ meet at the point C.

Fig-30-8
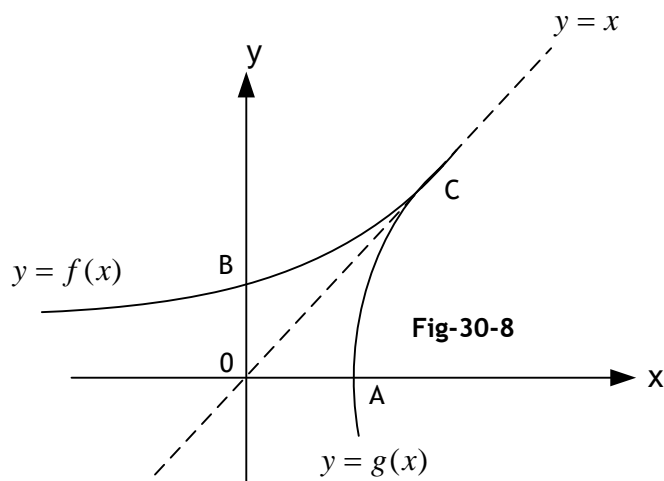
(ix) Find the exact coordinates of points A and B. Verify that the coordinates of C are (1, 1).

$$g(x)\Big|_{at\ A} = 0 = 1 + \ln x \qquad \Rightarrow \ln x = -1 \qquad \Rightarrow x = e^{-1} \qquad A(e^{-1}, 0)$$

$$f(x)\Big|_{at\ B} = e^{x-1}\Big|_{x=0} = e^{-1} \qquad \Rightarrow \quad B(0, e^{-1})$$

(x)　　Prove algebraically that $g(x)$ is the inverse of $f(x)$.

$y = f(x) = e^{x-1}$　　$\Rightarrow \ln y = (x-1)\ln e = x-1$　　$\Rightarrow x = 1 + \ln y$

$\Rightarrow f^{-1}(y) = 1 + \ln y$　　$\Rightarrow f^{-1}(x) = 1 + \ln x = g(x)$

(xi)　　Evaluate $\displaystyle\int_0^1 f(x)dx$, giving your answer in terms of e

$$\int_0^1 f(x)dx = \int_0^1 e^{x-1}dx$$

Let $u = x - 1$　　$\Rightarrow dx = du$

$$\int_0^1 f(x)dx = \int e^u du = e^u$$

$$\int_0^1 f(x)dx = \int_0^1 e^{x-1}dx = e^{x-1}\Big|_0^1 = e^{1-1} - e^{0-1} = e^0 - e^{-1} = 1 - \frac{1}{e}$$

(xii)　　Use integration by parts to find $\displaystyle\int \ln x\,dx$. Hence show that

$$\int_{e^{-1}}^1 g(x)dx = \frac{1}{e}$$

If　　$\displaystyle\int \ln x\,dx$

$\displaystyle\int \ln x\,dx = \int u\,dv = uv - \int v\,du$　　int *egration   by   part.*

Let　　$u = \ln x$　　*and*　　$dv = dx$

$du = \dfrac{dx}{x}$　　*and*　　$v = x$

$\displaystyle\int \ln x\,dx = \int u\,dv = uv - \int v\,du = x\ln x - \int x\frac{dx}{x} = x\ln x - \int dx = x\ln - x + C.$

Find the area of the region enclosed by the lines OA and OB, and the arcs AC and BC.

Ex-30-9: Fig-30-9 shows the curve $y = \dfrac{x^2}{3x-1}$. P is a turning point, and the curve has a vertical asymptote $x = a$.

Fig-30-9
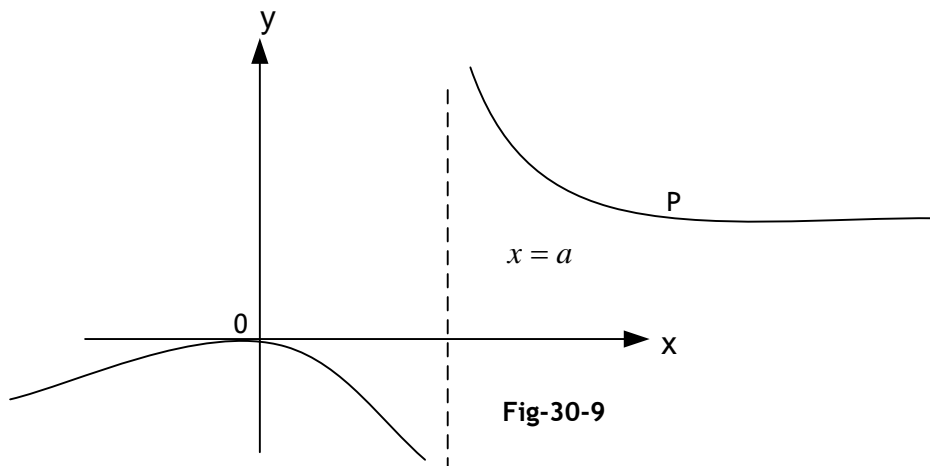
(i) Write down the value of a.

(ii) Show that $\dfrac{dy}{dx} = \dfrac{3(3x-2)}{(3x-1)^2}$.

(iii) Find the exact coordinates of the turning point P. Calculate the gradient of the curve when x=0.6 and x=0.8, and hence verify that P is a minimum point.

(iv) Using the substitution $u = 3x-1$, show that

$\int \dfrac{x^2}{3x-1}dx = \dfrac{1}{27}\int\left(u+2+\dfrac{1}{u}\right)du$. Hence find the exact area of the region

enclosed by the curve, the x-axis and the line $x = \dfrac{2}{3}$ and x=1.

PAPER-31

SECTION A (C3 May 2014)

Ex-31-1:

$$\int_0^{\frac{\pi}{6}}(1-\sin 3x)dx = \int_0^{\frac{\pi}{6}}dx - \int_0^{\frac{\pi}{6}}\sin 3xdx = x+\frac{1}{3}\cos 3x\Big|_0^{\frac{\pi}{6}} = \frac{\pi}{6}+\frac{1}{3}\cos\frac{\pi}{2}-\left(0+\frac{1}{3}\right) = \frac{\pi}{6}-\frac{1}{3}$$

Ex-31-2: $y = \ln(1 - \cos 2x)$

$$Gradient = \frac{dy}{dx}\bigg|_{x=\frac{\pi}{6}} = \frac{2\sin 2x}{1 - \cos 2x}\bigg|_{x=\frac{\pi}{6}} = \frac{2\sin\left(\frac{2\pi}{6}\right)}{1 - \cos\left(\frac{2\pi}{6}\right)} = \frac{2 \times \frac{\sqrt{3}}{2}}{1 - 0.5} = 2\sqrt{3}$$

Ex-31-3:   Solve the equation $|3 - 2x| = 4|x|$

$$3 - 2x = \pm 4x$$

(a) $3 - 2x = 4x \Rightarrow 6x = 3 \Rightarrow x = \dfrac{3}{6} = \dfrac{1}{2}$

(b) $3 - 2x = -4x \Rightarrow 2x = -3 \Rightarrow x = \dfrac{-3}{2} = -\dfrac{3}{2}$

Ex-31-4: $y = f(x) = a + \cos bx \qquad 0 \le x \le 2\pi$
$f(0) = a + \cos(0) = a + 1 = 3 \Rightarrow a = 3 - 1 = 2$

$$Gradient = \frac{dy}{dx}\bigg|_{x=0} = -b\sin bx\bigg|_{x=2\pi} = -b\sin(2\pi b) = 0 \Rightarrow 2\pi b = \pi \Rightarrow b = \frac{\pi}{2\pi} = \frac{1}{2}$$

$$y = 2 + \cos\frac{1}{2}x$$

$$x = 2 + \cos\frac{1}{2}y$$

$$\cos\frac{1}{2}y = x - 2$$

$$\frac{y}{2} = \cos^{-1}(x - 2)$$

$$y = f^{-1}(x) = 2\cos^{-1}(x - 2)$$

Ex-31-5: $V = \dfrac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \times \frac{dr}{dt} = 4\pi r^2 \times \frac{dr}{dt} \Rightarrow \frac{dV}{dt} \div 4\pi r^2 = \frac{1}{4\pi r^2} \times \frac{dV}{dt} = \frac{1}{4\pi(8)^2} \times 10 = \frac{10}{256\pi} = \frac{5}{128\pi}$$

Ex-31-6: $V = Ae^{-kt}$
$V = 20000\, e^{-0.2t}$

$$V\bigg|_{t=1} = 20000\, e^{-0.2t}\bigg|_{t=1} = 20000\, e^{-0.2} = \frac{20000}{e^{0.2}} = \frac{20000}{1.2214028} = 16374.615$$

$Loss = 20000 - 16374.615 = \pounds 3625.38$
$V = Ae^{-kt}$

$$13000 = 15000 \, e^{-kt} \bigg|_{t=1} = 15000 \, e^{-k} \Rightarrow e^{-k} = \frac{13000}{15000} = \frac{13}{15} \Rightarrow -k = \ln\left(\frac{13}{15}\right) = -0.1431 \Rightarrow k = 0.1431$$

$$15000 \, e^{-0.143 \, lt} = 20000 \, e^{-0.2t} \Rightarrow \frac{e^{-0.143 \, lt}}{e^{-0.2t}} = \frac{20000}{15000} = \frac{4}{3} \Rightarrow e^{0.0569t} = \frac{4}{3} \Rightarrow 0.0569 \, t = \ln\left(\frac{4}{3}\right) \Rightarrow t = 5.0559239 \cong 5$$

Ex-31-7: $\qquad y = f(x) = \dfrac{x}{\sqrt{2 + x^2}}$

$$f(-x) = \frac{-x}{\sqrt{2 + (-x)^2}} = \frac{-x}{\sqrt{2 + x^2}} = -f(x) \Rightarrow Odd$$

$$y = \frac{x}{\sqrt{2 + x^2}} = \frac{x}{\left(2 + x^2\right)^{\frac{1}{2}}} = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v \times \dfrac{du}{dx} - u \times \dfrac{dv}{dx}}{v^2} = \frac{\left(2 + x^2\right)^{\frac{1}{2}} \times 1 - x \times \dfrac{1}{2}\left(2 + x^2\right)^{-\frac{1}{2}} \times 2x}{\left(\left(2 + x^2\right)^{\frac{1}{2}}\right)^2} = \frac{\left(2 + x^2\right)^{\frac{1}{2}} - x^2\left(2 + x^2\right)^{-\frac{1}{2}}}{2 + x^2}$$

$$\frac{dy}{dx} = \frac{\left(2 + x^2\right)^{\frac{1}{2}} - x^2\left(2 + x^2\right)^{-\frac{1}{2}}}{2 + x^2} = \frac{a^{\frac{1}{2}} - x^2 a^{-\frac{1}{2}}}{a} = \frac{a^{-\frac{1}{2}}\left(a - x^2\right)}{a} = \frac{\left(a - x^2\right)}{a^{\frac{3}{2}}}$$

$$2 + x^2 = a \Rightarrow x^2 = a - 2$$

$$\frac{dy}{dx} = \frac{\left(a - a + 2\right)}{a^{\frac{3}{2}}} = \frac{2}{a^{\frac{3}{2}}} = \frac{2}{\left(2 + x^2\right)^{\frac{3}{2}}}$$

$$Gradient = \frac{dy}{dx} = \frac{\left(a - a + 2\right)}{a^{\frac{3}{2}}} = \frac{2}{a^{\frac{3}{2}}} = \frac{2}{\left(2 + x^2\right)^{\frac{3}{2}}}\bigg|_{x=0} = \frac{2}{2^{\frac{3}{2}}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$Area = \int_0^1 \frac{x}{\sqrt{2 + x^2}} dx$$

$$u = 2 + x^2 = du = 2dx \Rightarrow dx = \frac{du}{2}$$

$$Area = \int_0^1 \frac{x}{\sqrt{2 + x^2}} dx = \frac{1}{2}\int \frac{du}{u^{\frac{1}{2}}} = \frac{1}{2}\int u^{-\frac{1}{2}} du = \frac{2}{2} u^{\frac{1}{2}} = \sqrt{2 + x^2}\bigg|_0^1 = \sqrt{3} - \sqrt{2}$$

$$y = \frac{x}{\sqrt{2 + x^2}} \Rightarrow \frac{1}{y} = \frac{\sqrt{2 + x^2}}{x} \Rightarrow \frac{1}{y^2} = \frac{2 + x^2}{x^2} = \frac{x^2}{x^2} + \frac{2}{x^2} = 1 + \frac{2}{x^2}$$

$$\frac{1}{y^2} = 1 + \frac{2}{x^2} \Rightarrow y^{-2} = 1 + 2x^{-2} \Rightarrow -2y^{-3}\frac{dy}{dx} = -4x^{-3} \Rightarrow \frac{dy}{dx} = \frac{-4x^{-3}}{-2y^{-3}} = 2\frac{x^{-3}}{y^{-3}} = \frac{2y^3}{x^3}$$

$$\Rightarrow Gradient = \frac{dy}{dx} = \frac{2y^3}{x^3}\Big|_{x=0} = \infty (Math\ Error)$$

Ex-31-8: $y = xe^{-2x}$
$y = mx$

$$mx = xe^{-2x} \Rightarrow m = e^{-2x} \Rightarrow \ln m = -2x \Rightarrow x = -\frac{1}{2}\ln m \Rightarrow OK$$

$$y = xe^{-2x} = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = x(-2e^{-2x}) + e^{-2x} = e^{-2x}(1-2x)$$

$$Gradient = \frac{dy}{dx}\Big|_{x=-\frac{1}{2}\ln m} = e^{-2x}(1-2x)\Big|_{x=-\frac{1}{2}\ln m} = e^{-2\ln m^{\frac{1}{2}}}(1-2x)?$$

PAPER-32

SECTION A (C4-1-6-09)

Ex-32-1:   Express $4\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and

$$0 < \alpha < \frac{\pi}{2}$$

Hence solve the equation $4\cos\theta - \sin\theta = 3$ for $0 \le \theta \le 2\pi$

Ans:

$$4\cos\theta - \sin\theta = R\cos(\theta + \alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

$$4\cos\theta - \sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

$R\cos\alpha = 4$ ...(1)

$-R\sin\alpha = -1$ ...(2)

$R\sin\alpha = 1$ ...(2)

Dividing **Eq.2** by **Eq.1** $\tan\alpha = \dfrac{1}{4}$ $\Rightarrow \alpha = 14.04^0$

$R\cos 14.04^0 = 4 \Rightarrow R = \dfrac{4}{\cos 14.04^0} = 4.12$

$\Rightarrow 4\cos\theta - \sin\theta = R\cos(\theta+\alpha) = 4.12\cos(\theta+14.04^0)$

Hence solve the equation $4\cos\theta - \sin\theta = 3$ **for** $0 \le \theta \le 2\pi$

$4\cos\theta - \sin\theta = 4.12\cos(\theta+14.04^0) = 3 \quad \Rightarrow \cos(\theta+14.04^0) = \dfrac{3}{4.12} = 0.728$

$\Rightarrow \theta+14.04^0 = \cos^{-1}(0.728) = 43.27^0 \Rightarrow \theta = 43.27 - 14.04 = 29.23^0 \quad and \quad \theta = 360 - 29.23 = 389.23^0$

Ex-32-2: Using partial fraction, find $\displaystyle\int \dfrac{x}{(x+1)(2x+1)}dx$

Ans:

$$\dfrac{x}{(x+1)(2x+1)} = \dfrac{A}{x+1} + \dfrac{B}{2x+1} = \dfrac{A(2x+1)+B(x+1)}{(x+1)(2x+1)}$$

$$A(2x+1)+B(x+1) = x$$

$At \quad x = -1: \quad A(-2+1)+B(-1+1) = -1 \Rightarrow -A = -1 \Rightarrow A = 1$

$At \quad x = 0: \quad A(0+1)+B(0+1) = 0 \Rightarrow B = -A = -1$

$$\Rightarrow \dfrac{x}{(x+1)(2x+1)} = \dfrac{1}{x+1} - \dfrac{1}{2x+1}$$

$$\Rightarrow \int \dfrac{x}{(x+1)(2x+1)}dx = \int \dfrac{dx}{x+1} - \int \dfrac{dx}{2x+1} = \ln(x+1) - \dfrac{1}{2}\ln(2x+1) + c$$

Ex-32-3: A curve satisfies the differential equation $\dfrac{dy}{dx} = 3x^2 y$, and passes through the point (1, 1). Find y in terms of x.

$$\dfrac{dy}{dx} = 3x^2 y \quad \Rightarrow \dfrac{dy}{y} = 3x^2 dx$$

$$\Rightarrow \int \dfrac{dy}{y} = \int 3x^2 dx \quad \Rightarrow \ln y = x^3 + c$$

$$\ln y = x^3 + c \Rightarrow \ln(1) = 1^3 + c \Rightarrow c = -1$$

$$\ln y = x^3 - 1 \Rightarrow y = e^{(x^3-1)}$$

Ex-32-4: The part of the curve $y = 4 - x^2$ that is above the x-axis is rotated about the y-axis. This is shown in Fig-32-4.

Find the volume of revolution produced, giving your answer in terms of π
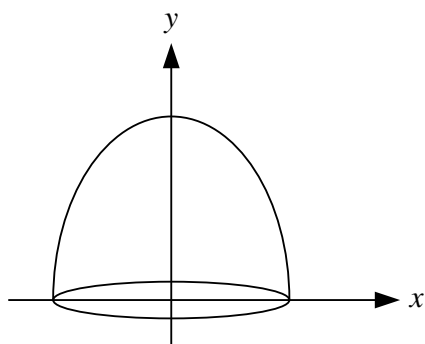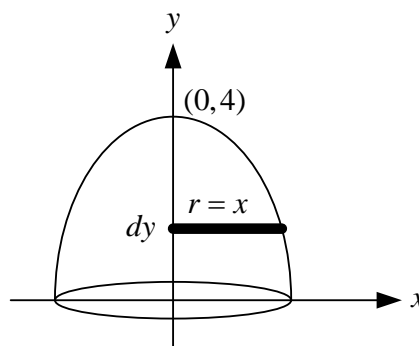
Fig-32-4

Fig-32-4

$$V = \int \pi r^2 \, dy = \int \pi x^2 \, dy = \pi \int_0^4 (4 - y) \, dy = \pi \left(4y - \frac{y^2}{2} \right) \Big|_0^4 = \pi(16 - 8) = 8\pi$$

Ex-32-5: A curve has parametric equations

$$x = at^3, \qquad\qquad y = \frac{a}{1 + t^2} \qquad\qquad \text{where } a \text{ is a constant.}$$

Show that $\dfrac{dy}{dx} = \dfrac{-2}{3t\left(1 + t^2\right)^2}$

Hence find the gradient of the curve at the point $\left(a, \dfrac{1}{2}a \right)$

Ans:

$$x = at^3, \qquad\qquad y = \frac{a}{1 + t^2}$$

$$\frac{dx}{dt} = 3at^2, \qquad\qquad \frac{dy}{dt} = \frac{-2at}{\left(1 + t^2\right)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad \text{Remember that} \quad \frac{dt}{dx} = \frac{1}{\dfrac{dx}{dt}} = \frac{1}{3at^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{-2at}{\left(1 + t^2\right)^2}\left(\frac{1}{3at^2} \right) = -\frac{2}{3t\left(1 + t^2\right)^2}$$

Hence find the gradient of the curve at the point $\left(a, \dfrac{1}{2}a\right)$

$$x = at^3 \Rightarrow a = at^3 \Rightarrow t = 1$$

$$y = \frac{a}{1+t^2} \Rightarrow \frac{1}{2}a = \frac{a}{1+t^2} \Rightarrow 1+t^2 = 2 \Rightarrow t = 1$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t=1} = \frac{-2}{3t\left(1+t^2\right)^2} = -\frac{2}{3(1+1)^2} = -\frac{2}{12} = -\frac{1}{6}$$

Ex-32-6: Given that $\cos ec^2\theta - \cot\theta = 3$, show that $\cot^2\theta - \cot\theta - 2 = 0$ for $0 \le \theta \le 2\pi$

Note: Use the following to remind yourself

$$\sin^2\theta + \cos^2\theta = 1 \qquad\qquad ....(1)$$

Divide each side by $\sin^2\theta$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \Rightarrow 1 + \cot^2\theta = \cos ec^2\theta$$

Divide each side by $\cos^2\theta$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \Rightarrow \tan^2\theta + 1 = \sec^2\theta$$

Ans:

$$\cos ec^2\theta = 1 + \cot^2\theta$$

$$\Rightarrow 1 + \cot^2\theta - \cot\theta - 3 = 0$$

$$\Rightarrow \cot^2\theta - \cot\theta - 2 = 0$$

Hence solve the equation $\cos ec^2\theta - \cot\theta = 3$ **for** $0 \le \theta \le 180^0$

$$\cot^2\theta - \cot\theta - 2 = 0 \Rightarrow (\cot\theta - 2)(\cot\theta + 1) = 0$$

If $\cot\theta - 2 = 0 \Rightarrow \cot\theta = 2 \Rightarrow \theta = \tan^{-1}(2) = 63.43^0, 243.43^0$

If $\cot\theta + 1 = 0 \Rightarrow \cot\theta = -1 \Rightarrow \theta = \tan^{-1}(-1) = 315^0, 225^0$

Ex-32-7: When a light ray passes from air to glass, it is deflected through an angle. The light ray ABC starts at point A(1, 2, 2), and enters a glass object at point B(0, 0, 2). The surface of the glass object is a plane with normal vector n. Fig-32-7 shows a cross-section of the glass object in the plane of the light ray and n.

Fig.7
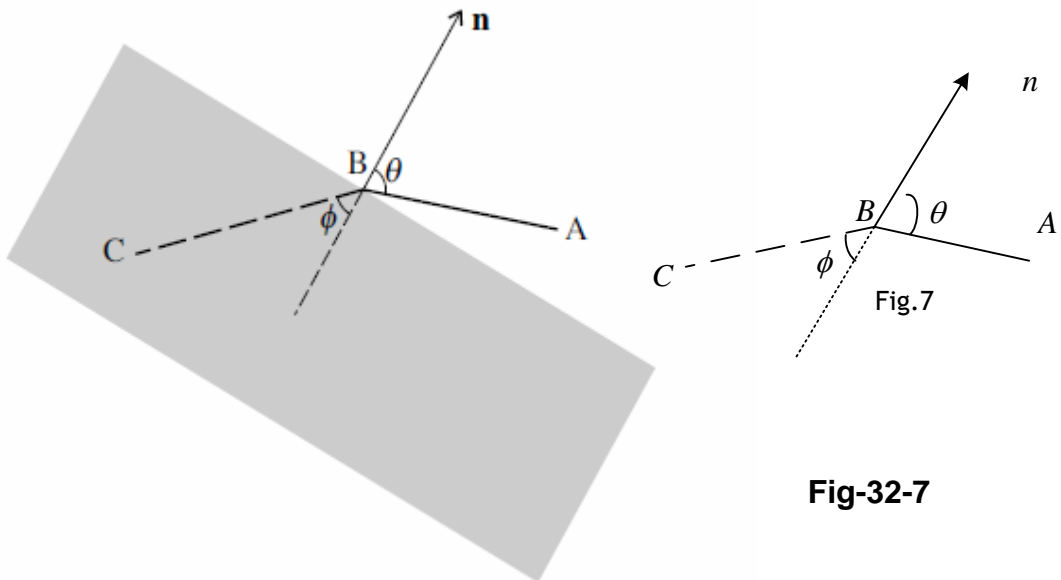
Fig-32-7

(i) Find the vector $\overrightarrow{AB}$ and a vector equation of the line AB.

Ans:

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}$$

$$r = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} \quad \textbf{This is correct}$$

$$\frac{x-1}{-1} = \frac{y-2}{-2} \Rightarrow -y+2 = -2x+2 \Rightarrow y = 2x$$

OR $\quad r = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad$ **This is also correct**

$$\frac{x-0}{1} = \frac{y-0}{2} \Rightarrow y = 2x \text{ , } \textbf{therefore both of them the same}$$

The surface of the glass object is a plane with equation $x + z = 2$. AB makes an acute angle θ with the normal to this plane.

(ii)   Write down the normal vector n, and hence calculate θ, giving your answer in degrees.

Note: If the normal to the plane is $\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = n_1 i + n_2 j + n_3 k$, then the equation of

the plane is: $n_1 x + n_2 y + n_3 z + d = 0$, where $d = -\vec{a}.\vec{n}$ and $\vec{a}$ is the position vector of a point on the plane

$$\Rightarrow n = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Note: Angle between $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$ is θ, where

$$\cos\theta = \frac{\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} d \\ e \\ f \end{pmatrix}}{\sqrt{a^2 + b^2 + c^2} \times \sqrt{d^2 + e^2 + f^2}}$$

$$\Rightarrow n = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \vec{AB} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} \qquad \vec{BA} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$
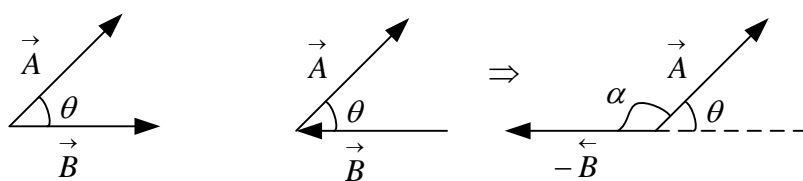
$$\cos\theta = \frac{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}{\sqrt{1^2 + 0^2 + 1^2} \times \sqrt{(-1)^2 + (-2)^2 + 0^2}} = \frac{1}{\sqrt{2}\sqrt{5}} = \frac{1}{10} \Rightarrow \theta = 71.57^0$$

The line BC has vector equation $r = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}$. This line makes an acute angle

ϕ with the normal to the plane

(iii) Show that $\phi = 45^0$.

$\Rightarrow n = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ because it is in the opposite direction. $\qquad r = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}$

$$\cos \phi = \frac{\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}}{\sqrt{1^2 + 0^2 + (-1)^2} \times \sqrt{(-2)^2 + (-2)^2 + (-1)^2}} = \frac{2+1}{\sqrt{2}\sqrt{9}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45^0$$

(iv) Snell's Law states that $\sin \theta = k \sin \phi$, where k is a constant called the refractive index. Find k.

Ans:

$$\sin \theta = k \sin \phi \Rightarrow k = \frac{\sin \theta}{\sin \phi} = \frac{\sin 71.57^0}{\sin 45^0} = \frac{0.9487}{0.7071} = 1.34$$

$$\Rightarrow k = 1.34$$

The light ray leaves the glass object through a plane with equation $x + z = -1$. Units are centimetres.

(v) Find the point of intersection of the line BC with the plane $x + z = -1$, hence find the distance the light ray travels through the glass object.

Ans:

The line BC has vector equation $r = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}$.

The line BC has vector equation $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix}$.

$x = -2\mu \qquad y = -2\mu \qquad z = 2 - \mu$ and also $x + z = -1$ from the question

$-2\mu + 2 - \mu = -1 \Rightarrow -3\mu = -3 \Rightarrow \mu = 1$

Therefore, point of intersection is: $r = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$,

the point is: (-2, -2, 1)

Ex-32-8: Archimedes, about 2200 years ago, used regular polygons inside and

circles to obtain approximations for π.

(i) Fig-32-8-1 shows a regular 12-sided polygon inscribed in a circle of radius 1 unit, centre O. AB is one of the sides of the polygon. C is the midpoint of AB. Archimedes used the fact that the circumference of the circle is greater than the perimeter of this polygon.

Fig. 8.1   Fig-32-8-1

$AC = \dfrac{AB}{2}$

$\theta = \dfrac{1}{2}\left(\dfrac{360}{12}\right) = 15^0$

(A) Show that $AB = 2\sin 15^0$.
Ans:

$\sin\theta = \sin 15^0 = \dfrac{AC}{1} = \dfrac{AB}{2} \Rightarrow AB = 2\sin 15^0$

(B) Use a double angle formula to express $\cos 30^0$ in terms of $\sin 15^0$. Using the exact value of $\cos 30^0$, show that $\sin 15^0 = \dfrac{1}{2}\sqrt{2 - \sqrt{3}}$

Ans:

$\cos(30^0) = \cos(15^0 + 15^0) = \cos 15^0 \times \cos 15^0 - \sin 15^0 \times \sin 15^0 = \cos^2 15^0 - \sin^2 15$

$\Rightarrow \cos(30^0) = \cos^2 15^0 - \sin^2 15 = 1 - 2\sin^2 15^0$

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$$\Rightarrow \cos(30^0) = 1 - 2\sin^2 15^0 \Rightarrow 2\sin^2 15^0 = 1 - \cos 30^0 \Rightarrow \sin 15^0 = \sqrt{\frac{1 - \cos 30^0}{2}} = ?$$

$$\Rightarrow \cos(30^0) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \sin 15^0 = \sqrt{\frac{1 - \cos 30^0}{2}} = \sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2 - \sqrt{3}}{2}}{2}} = \frac{1}{2}\sqrt{2 - \sqrt{3}}$$

(C) Use this result to find an exact expression for the perimeter of the polygon.
Ans:

$$AB = 2\sin 15^0 = 2\left(\frac{1}{2}\sqrt{2 - \sqrt{3}}\right) = \sqrt{2 - \sqrt{3}}$$

Perimeter= $12AB = 12\sqrt{2 - \sqrt{3}}$

Hence show that $\pi > 6\sqrt{2 - \sqrt{3}}$

Ans: *Perimeter of the circle > perimeter of polygon* **(see Fig.8)**

$$2\pi r = 2\pi > 12\sqrt{2 - \sqrt{3}} \Rightarrow \pi > 6\sqrt{2 - \sqrt{3}}$$

(ii) In Fig.8.2, a regular 12-sided polygon lies outside the circle of radius 1 unit, which touches each side of the polygon. F is the midpoint of DE. Archimedes used the fact that the circumference of the circle is less than the perimeter of this polygon.
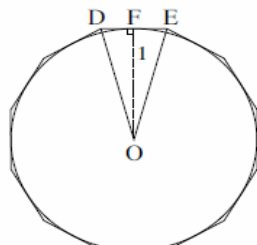
Fig. 8.2
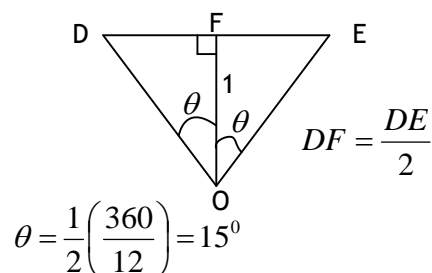
$$\theta = \frac{1}{2}\left(\frac{360}{12}\right) = 15^0$$

$$DF = \frac{DE}{2}$$

(A) Show that $DE = 2\tan 15^0$.

Ans:

$$\tan\theta = \tan 15^0 = \frac{FE}{1} = FE = \frac{DE}{2} \Rightarrow DE = 2\tan 15^0$$

(B) Let $t = \tan 15^0$. Use a double angle formula to express $\tan 30^0$ in terms of t. Hence show that $t^2 + 2\sqrt{3}t - 1 = 0$

Ans:

$$\tan 30^0 = \tan\left(15^0 + 15^0\right) = \frac{\tan 15^0 + \tan 15^0}{1 - \tan 15^0 \tan 15^0} = \frac{t+t}{1-t\times t} = \frac{2t}{1-t^2}$$

$$\tan 30^0 = \frac{2t}{1-t^2} = \frac{\sin 30^0}{\cos 30^0} = \frac{\dfrac{1}{2}}{\dfrac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} \Rightarrow 1-t^2 = 2\sqrt{3}\,t \Rightarrow t^2 + 2\sqrt{3}\,t - 1 = 0$$

(C) Solve this equation, and hence show that $\pi < 12\left(2 - \sqrt{3}\right)$

Ans: *Perimeter of the circle < perimeter of polygon* **(see Fig.8.2)**

$$2\pi r = 2\pi < 12 DE = 24\tan 15^0 = 24t$$

$$t^2 + 2\sqrt{3}\,t - 1 = 0$$

$$t = \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2} = \frac{-2\sqrt{3} \pm \sqrt{16}}{2} = \frac{-2\sqrt{3} \pm 4}{2} = -\sqrt{3} \pm 2$$

$$t = -\sqrt{3} \pm 2$$
$$\Rightarrow t = 2 - \sqrt{3}$$

Circumference < Perimeter

$$\Rightarrow 2\pi < 24\left(2 - \sqrt{3}\right)$$
$$\Rightarrow \pi < 12\left(2 - \sqrt{3}\right)$$

$$t^2 + 2\sqrt{3}\,t - 1 = 0$$

(iii) Use the results in parts (i) (C) and (ii) (C) to establish upper and lower bounds for the value of π, giving your answers in decimal form.

Ans:

$$6\left(2-\sqrt{3}\right) < \pi < 12\left(2-\sqrt{3}\right)$$

$$\Rightarrow 3.106 < \pi < 3.215$$

PAPER-33

SECTION A (C4-1-1-09)

Ex-33-1: Express $\dfrac{3x+2}{x(x^2+1)}$ in partial fraction.

Ans:

$$\frac{3x+2}{x\left(x^2+1\right)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$A = \left.\frac{3x+2}{x^2+1}\right|_{x=0} = \frac{2}{1} = 2$$

$$\frac{3x+2}{x\left(x^2+1\right)} = \frac{2}{x} + \frac{Bx+C}{x^2+1} = \frac{2x^2+2+Bx^2+Cx}{x(x^2+1)}$$

$$3x+2 = 2x^2+2+Bx^2+Cx$$

When $x=1$ $\Rightarrow 3+2 = 2+2+B+C$ $\Rightarrow B+C = 1$

When $x=-1$ $\Rightarrow -3+2 = 2+2+B-C$ $\Rightarrow B-C = -5$

$\Rightarrow 2B = -4$ $\Rightarrow B = -2$ $\quad and \quad C = 1-B = 1+2 = 3$

$$\frac{3x+2}{x\left(x^2+1\right)} = \frac{2}{x} + \frac{3-2x}{x^2+1}$$

Ex-33-2: Show that $(1+2x)^{\frac{1}{3}} = 1 + \dfrac{2}{3}x - \dfrac{4}{9}x^2 + \ ...$, and find the next term in the expansion. State the set of values of x for which the expansion is valid.

Note: Use the general case.

Ans: $(a+b)^n = \dfrac{a^n}{0!} + \dfrac{n \times a^{n-1}b}{1!} + \dfrac{n(n-1) \times a^{n-2}b^2}{2!} + \dfrac{n(n-1)(n-2) \times a^{n-3}b^3}{3!} + ... + b^n$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 \ldots,$$

$$(1+2x)^{\frac{1}{3}} = 1 + \left(\frac{1}{3}\right)(2x) + \frac{\frac{1}{3}\times\left(-\frac{2}{3}\right)(2x)^2}{2!} + \frac{\frac{1}{3}\times\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(2x)^3}{3!} + \ldots$$

$$(1+2x)^{\frac{1}{3}} = 1 + \frac{2}{3}x - \frac{2}{9}\left(\frac{4x^2}{2}\right) + \ldots$$

$$\Rightarrow (1+2x)^{\frac{1}{3}} = 1 + \frac{2}{3}x - \frac{4}{9}x^2 + \ldots$$

The next term is: $\dfrac{\frac{1}{3}\times\left(-\frac{2}{3}\right)\times\left(-\frac{5}{3}\right)(2x)^3}{3!} = \dfrac{80x^3}{27\times6} = \dfrac{40x^3}{81}$

OR

Note: **The following can be used when n is a positive integer!!**

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots$$

$$^nC_k = \binom{n}{k} = \frac{n!}{(n-k)!\,k!} \qquad \text{and} \qquad \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Note: General case:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 \ldots,$$

Note: Learn this

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 \ldots,$$

$$(1+2x)^{\frac{1}{3}} = 1 + \frac{1}{3}(2x) + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}(2x)^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(2x)^3 + \ldots,$$

$$\left(1+2x\right)^{\frac{1}{3}} = 1 + \frac{2}{3}x - \frac{4}{9}x^2 + \frac{40}{81}x^3 + ...,$$

Valid for $|2x| < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$

Ex-33-3: Vectors a and b are given by $a = 2i + j - k$ and $b = 4i - 2j + k$

Find constants λ and μ such that $\lambda a + \mu b = 4j - 3k$

$\lambda a + \mu b = 4j - 3k$

$\lambda(2i + j - k) + \mu(4i - 2j + k) = 4j - 3k$

$(2\lambda + 4\mu)i + (\lambda - 2\mu)j - (\lambda + \mu)k = 4j - 3k$

$2\lambda + 4\mu = 0$ ...(1)

$\lambda - 2\mu = 4$ ...(2)

$2\lambda - 4\mu = 8$ ...(2)

Adding Eq.1 and 2

$4\lambda = 8 \Rightarrow \lambda = 2$

$4\mu = -2\lambda = -4 \quad \Rightarrow \mu = -1$

Ex-33-4: Prove that $\cot\beta - \cot\alpha = \dfrac{\sin(\alpha - \beta)}{\sin\alpha\sin\beta}$

$$\cot\beta - \cot\alpha = \frac{\cos\beta}{\sin\beta} - \frac{\cos\alpha}{\sin\alpha} = \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\beta\sin\alpha} = \frac{\sin(\alpha - \beta)}{\sin\alpha\sin\beta}$$

Ex-33-5: (i)   Write down normal vectors to the planes $2x - y + z = 2$ and

$x - z = 1$. Hence find the acute angle between the planes.

Ans:

(i)

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Angle between planes is θ, where

$$\cos\theta = \frac{2\times 1 + (-1)\times 0 + 1\times(-1)}{\sqrt{2^2 + (-1)^2 + 1^2}\ \sqrt{1^2 + 0^2 + (-1)^2}} = \frac{1}{\sqrt{12}}$$

$$\Rightarrow \theta = 73.2^0 \text{ or } \Rightarrow \theta = 1.28\ rads$$

(ii) Write down a vector equation of the line through (2, 0, 1) perpendicular to the plane $2x - y + z = 2$.

Find the point of intersection of this line with the plane.

(i)

$$r = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 + 2\lambda \\ -\lambda \\ 1 + \lambda \end{pmatrix}$$

$$\Rightarrow 2(2 + 2\lambda) - (-\lambda) + (1 + \lambda) = 2$$
$$\Rightarrow 5 + 6\lambda = 2$$
$$\Rightarrow \lambda = -\frac{1}{2}$$

So point of intersection is $\left(1, \dfrac{1}{2}, \dfrac{1}{2}\right)$

Ex-33-6:   (i)   Express $\cos\theta + \sqrt{3}\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$

and α is acute, expressing α in terms of π

Ans: $\cos\theta + \sqrt{3}\sin\theta = R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$\cos\theta + \sqrt{3}\sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

$R\cos\alpha = 1$ ...(1)

$R\sin\alpha = \sqrt{3}$ ...(2)

Dividing Eq.2 and Eq.1 $\qquad \tan\alpha = \sqrt{3} \qquad \Rightarrow \alpha = 60^0$

$$R\cos 60^0 = 1 \Rightarrow R = \frac{1}{\cos 60^0} = 2$$

$$\Rightarrow \cos\theta + \sqrt{3}\sin\theta = R\cos(\theta - \alpha) = 2\cos(\theta - 60^0) = 2\cos\left(\theta - \frac{\pi}{3}\right)$$

(ii)   Write down the derivative of $\tan\theta$.

$$\frac{d}{d\theta}(\tan\theta) = \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right) = \frac{\cos\theta\cos\theta + \sin\theta\sin\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$$

Hence show that $\displaystyle\int_0^{\frac{\pi}{3}} \frac{1}{\left(\cos\theta + \sqrt{3}\sin\theta\right)^2}d\theta = \frac{\sqrt{3}}{4}$

$$\int_0^{\frac{\pi}{3}} \frac{1}{\left(\cos\theta + \sqrt{3}\sin\theta\right)^2}d\theta = \int_0^{\frac{\pi}{3}} \left(\frac{d\theta}{\left(2\cos\left(\theta - \frac{\pi}{3}\right)\right)^2}\right) = \frac{1}{4}\int_0^{\frac{\pi}{3}} \frac{d\theta}{\cos^2\left(\theta - \frac{\pi}{3}\right)} = \frac{1}{4}\int_0^{\frac{\pi}{3}} \sec^2\left(\theta - \frac{\pi}{3}\right)d\theta$$

$$\frac{1}{4}\int_0^{\frac{\pi}{3}} \sec^2\left(\theta - \frac{\pi}{3}\right)d\theta = \frac{1}{4}\tan\left(\theta - \frac{\pi}{3}\right)\Big|_0^{\frac{\pi}{3}} = \frac{1}{4}\tan\left(\frac{\pi}{3} - \frac{\pi}{3}\right) - \frac{1}{4}\tan\left(0 - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{4}$$

SECTION B(P-33)

Ex-33-7:   Scientists can estimate the time elapsed since an animal died by measuring its body temperature.

(i)   Assuming the temperature goes down at a constant rate of 1.5 degrees Fahrenheit per hour, estimate how long it will take for the temperature to drop.

(A)   from $98^0 F$ to $89^0 F$

(B)   from $98^0 F$ to $80^0 F$

Ans:

(A) $\dfrac{9}{1.5} = 6$ *hours*

(B) $\dfrac{18}{1.5} = 12$ *hours*

In practice, rate of temperature loss is not likely to be constant. A better model is provided by Newton's law of cooling, which states that the temperature θ in degrees Fahrenheit t hours after death is given by the differential equation.

$$\frac{d\theta}{dt} = -k\left(\theta - \theta_0\right)$$

Where $\theta_0{}^0 F$ is the air temperature and k is a constant.

(ii)   Show by integration that the solution of this equation is

$\theta = \theta_0 + Ae^{-kt}$, where A is a constant.

Ans:

$$\frac{d\theta}{dt} = -k\left(\theta - \theta_0\right)$$

$$\int \frac{d\theta}{\theta - \theta_0} = \int -kdt$$

$$\ln\left(\theta - \theta_0\right) = kt + c$$

$$\theta = \theta_0 + Ae^{-kt}$$

The value of θ_0 is 50, and the initial value of θ is 98. The initial rate of temperature is $1.5^0\ F$ per hour.

(iii)   Find A, and show that $k = 0.03125$
Ans:

$$98 = 50 + Ae^0$$
$$\Rightarrow A = 48$$

Initially $\dfrac{d\theta}{dt} = -k\left(98 - 50\right) = -48k = -1.5$

$$\Rightarrow k = 0.03125$$

(iv)   Use this model to calculate how long it will take for the temperature to drop
(A)   from $98^0 F$ to $89^0 F$

(B)   from $98^0 F$ to $80^0 F$

Ans:

$$98 = 50 + 48e^{-0.03125t}$$

(A) $\Rightarrow \dfrac{39}{48} = e^{-0.03125t}$

$\Rightarrow t = \ln\left(\dfrac{39}{48}\right) / (-0.03125) = 6.64 \ hrs$

$$80 = 50 + 48e^{-0.03125t}$$

(B) $\Rightarrow \dfrac{30}{48} = e^{-0.03125t}$

$\Rightarrow t = \ln\left(\dfrac{30}{48}\right) / (-0.03125) = 15 \ hrs$

(v)   Comment on the results obtained in part (i) and (iv).

Ans:

Models disagree more for greater temperature loss

Ex-33-8:   Fig.8 illustrates a hot air balloon on its side. The balloon is modelled by the of revolution about the x-axis of the curve with parametric equations

$$x = 2 + 2\sin\theta \qquad\qquad y = 2\cos\theta + \sin 2\theta \quad (0 \le \theta \le 2\pi)$$

The curve crosses the x-axis at the point A(4, 0). B and C are maximum and minimum points on the curve. Units on the axes are metres.

(i)   Find $\dfrac{dy}{dx}$ in terms of θ.

Ans:

$$\dfrac{dy}{dx} = 2\cos 2\theta - 2\sin\theta, \quad \dfrac{dx}{d\theta} = 2\cos\theta$$

$$\dfrac{dy}{dx} = \dfrac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}}$$

$$\dfrac{dy}{dx} = \dfrac{2\cos 2\theta - 2\sin\theta}{2\cos\theta} = \dfrac{\cos 2\theta - \sin\theta}{\cos\theta}$$

(ii)   Verify that $\dfrac{dy}{dx} = 0$ when $\theta = \dfrac{\pi}{6}$, and find the exact coordinates of B. Hence find the maximum width BC of the balloon.
Ans:

When $\theta = \dfrac{\pi}{6}$, $\dfrac{dy}{dx} = \dfrac{\cos\dfrac{\pi}{3} - \sin\dfrac{\pi}{6}}{\cos\dfrac{\pi}{6}} = \dfrac{\dfrac{1}{2} - \dfrac{1}{2}}{\dfrac{\sqrt{3}}{2}} = 0$

Coords of B:

$$x = 2 + 2\sin\left(\dfrac{\pi}{6}\right) = 3$$

$$y = 2\cos\left(\dfrac{\pi}{6}\right) + \sin\left(\dfrac{\pi}{3}\right) = \dfrac{3\sqrt{3}}{2}$$

$$BC = 2 \times \dfrac{3\sqrt{3}}{2} = 3\sqrt{3}$$

(iii)　(A)　Show that $y = x\cos\theta$

Ans:

$$\begin{aligned} y &= 2\cos\theta + \sin 2\theta \\ &= 2\cos\theta + 2\sin\theta\cos\theta \\ &= 2\cos\theta(1 + \sin\theta) \\ &= (2 + 2\sin\theta)\cos\theta \\ &= x\cos\theta \end{aligned}$$

(B)　Find $\sin\theta$ in terms of x and show that $\cos^2\theta = x - \dfrac{1}{4}x^2$

Ans:

$$x = 2 + 2\sin\theta$$

$$\sin\theta = \dfrac{1}{2}(x - 2)$$

$$\begin{aligned} \cos^2\theta &= 1 - \sin^2\theta \\ &= 1 - \dfrac{1}{4}(x - 2)^2 \\ &= 1 - \dfrac{1}{4}x^2 + x - 1 \\ &= \left(x - \dfrac{1}{4}x^2\right) \end{aligned}$$

(C)　Hence show that the Cartesian equation of the curve is $y^2 = x^3 - \dfrac{1}{4}x^4$

Ans:

Cartesian equation is $y^2 = x^2 \cos^2 \theta = x^2 \left(x - \dfrac{1}{4} x^2 \right) = x^3 - \dfrac{1}{4} x^4$

(iv)   Find the volume of the balloon.

Ans:

$$V = \int_0^4 \pi y^2 dx = \pi \int_0^4 \left(x^3 - \dfrac{1}{4} x^4 \right) dx$$

$$\pi \left[\dfrac{1}{4} x^4 - \dfrac{1}{20} x^5 \right]\Big|_0^4 = \begin{cases} \pi(64 - 51.2) \\[2mm] 12.8\pi = 40.2\,(m^3) \end{cases}$$

PAPER-34

Ex-34-1:   Express $\dfrac{x}{x^2 - 4} + \dfrac{2}{x + 2}$ as a single fraction, simplifying your answer.

$$\dfrac{x}{x^2 - 4} + \dfrac{2}{x + 2} = \dfrac{x}{(x+2)(x-2)} + \dfrac{2}{x+2} = \dfrac{x + 2(x-2)}{(x+2)(x-2)} = \dfrac{x + 2x - 4}{(x+2)(x-2)} = \dfrac{3x - 4}{x^2 - 4}$$

Ex-34-2:   Fig-34-2 shows the curve $y = \sqrt{1 + e^{2x}}$.

Fig-34-2
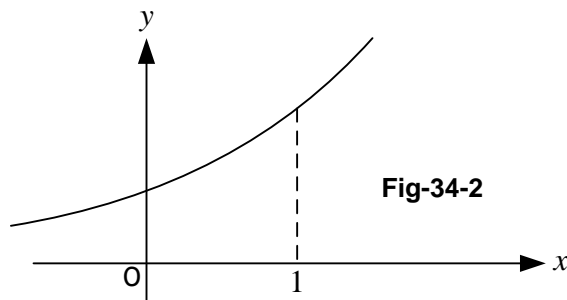
The region bounded by the curve, the *x*-axis, the *y*-axis and the line *x* = 1 is rotated through 360^0 about the *x*-axis.

Show that the volume of the solid of revolution produced is $\dfrac{1}{2} \pi \left(1 + e^2 \right)$.

Ans:

$$\int_0^1 \pi r^2 dx = \pi \int_0^1 \left(\sqrt{1+e^{2x}}\right)^2 dx = \pi \int_0^1 (1+e^{2x})dx = \pi \left[x + \frac{1}{2}e^{2x}\right]_0^1 = \pi \left[\left(1 + \frac{1}{2}e^2\right) - \left(0 + \frac{1}{2}\right)\right] = \frac{\pi}{2}(1+e^2)$$

Ex-34-3: Solve the equation $\cos 2\theta = \sin \theta$ for $0 \le \theta \le 2\pi$, giving your answers in terms of π.

Ans:

$$\cos 2\theta = \cos(\theta + \theta) = \cos\theta \cos\theta - \sin\theta \sin\theta = \cos^2\theta - \sin^2\theta = 1 - \sin^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$
$$\Rightarrow 1 - 2\sin^2\theta = \sin\theta \quad \Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0 \quad \Rightarrow (2\sin\theta - 1)(\sin\theta + 1) = 0$$

$$\Rightarrow (2\sin\theta - 1)(\sin\theta + 1) = 0$$

If $(2\sin\theta - 1) = 0 \quad \Rightarrow \sin\theta = \frac{1}{2} \quad \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

If $(\sin\theta + 1) = 0 \quad \Rightarrow \sin\theta = -1 \quad \Rightarrow \theta = -\frac{\pi}{2}, \quad \Rightarrow \theta = \frac{3\pi}{2}$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$$

Ex-34-4: Given that $x = 2\sec\theta$ and $y = 3\tan\theta$, show that $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

Ans:

$$x = 2\sec\theta \Rightarrow x^2 = 4\sec^2\theta$$

$$y = 3\tan\theta \Rightarrow y^2 = 9\tan^2\theta$$

$$\Rightarrow \frac{x^2}{4} - \frac{y^2}{9} = \frac{4\sec^2\theta}{4} - \frac{9\tan^2\theta}{9} = \sec^2\theta - \tan^2\theta = 1 + \tan^2\theta - \tan^2\theta = 1$$

Ex-34-5: A curve has parametric equations $x = 1 + u^2$, $y = 2u^3$.

    (i) Find $\dfrac{dy}{dx}$ in terms of u.

    Ans:

$$x = 1 + u^2 \implies \frac{dx}{du} = 2u$$

$$y = 2u^3 \implies \frac{dy}{du} = 6u^2$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 6u^2 \times \frac{1}{2u} = 3u$$

(ii) Hence find the gradient of the curve at the point with coordinates (5, 16).

Ans:

$$gradient = \frac{dy}{dx} = 3u$$

$$x = 1 + u^2 \implies 5 = 1 + u^2 \implies u^2 = 4 \implies u = \pm\sqrt{4} = \pm 2$$

OR

$$y = 2u^3 \implies 16 = 2u^3 \implies u^3 = 8 \implies u = 2$$

$$\implies gradient = \frac{dy}{dx} = 3u = 3(2) = 6$$

Ex-34-6: **(i)** Find the first three non-zero terms of the binomial series expansion of $\dfrac{1}{\sqrt{1+4x^2}}$, and state the set of values of x for which the expansion is valid.

Ans:

$$(1 + zx)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1 \times 2 \times 3 \times 4 \ldots \times r}x^r$$

$$\frac{1}{\sqrt{1+4x^2}} = \frac{1}{(1+4x^2)^{\frac{1}{2}}} = (1+4x^2)^{-\frac{1}{2}} = 1 - \frac{1}{2}(4x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2}(4x^2)^2 + \ldots = 1 - 2x^2 + 6x^4 + \ldots$$

Valid for $\left|4x^2\right| < 1 \implies -\frac{1}{2} < x < \frac{1}{2}$

(ii)   Hence find the first three non-zero terms of the series expansion of

$$\frac{1-x^2}{\sqrt{1+4x^2}}$$

$$\frac{1-x^2}{\sqrt{1+4x^2}} = \left(1-x^2\right)\left(1-2x^2+6x^4+...\right) = 1-3x^2+8x^4+...$$

Ex-34-7:   Express $\sqrt{3}\sin x - \cos x$ in the form $R\cos(x-\alpha)$, where $R>0$

and $0 < \alpha < \dfrac{\pi}{2}$ .Expressing α in terms of $k\pi$

Find the exact coordinates of the maximum point of the curve $y = \sqrt{3}\sin x - \cos x$
for which 0 < x < 2π.

Ans:

$$\sqrt{3}\sin x - \cos x = R\cos(x-\alpha) = R\cos x\cos\alpha + R\sin x\sin\alpha$$

$$R\sin\alpha = \sqrt{3} \qquad\qquad ...(1)$$

$$R\cos\alpha = -1 \qquad\qquad ...(2)$$

Dividing Eq.1 by Eq.2 $\tan\alpha = -\sqrt{3} \qquad \Rightarrow \alpha = -60^0$

$$R\sin(-60^0) = \sqrt{3} \Rightarrow R = \frac{\sqrt{3}}{\sin(-60^0)} = -2$$

$$\sqrt{3}\sin x - \cos x = R\cos(x-\alpha) = -2\cos\left(x+60^0\right) = -2\cos\left(x+\frac{\pi}{3}\right)$$

$$\sqrt{3}\sin x - \cos x = -2\cos\left(x+\frac{\pi}{3}\right)\Big|_{\text{max when}} \quad x+\frac{\pi}{3} = \pi \Rightarrow x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

SECTION B (P-34)

Ex-34-8:   The upper and lower surfaces of a coal seam are modelled as planes ABC
and DEF, as shown in Fig-34-8. All dimensions are metres.

Fig-34-8
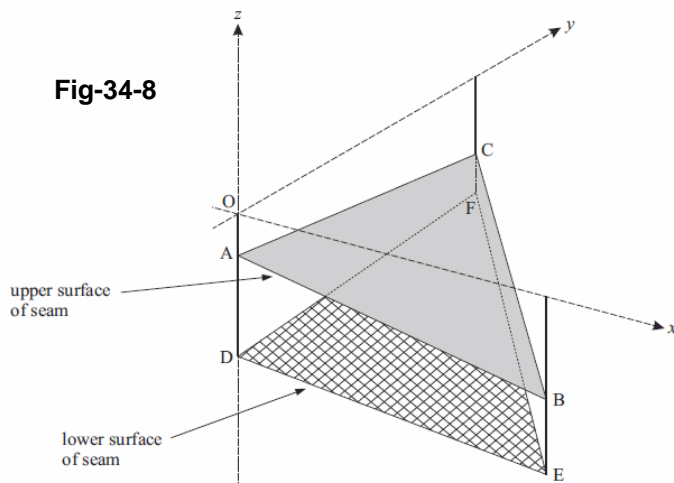
Relative to axes Ox (due east), Oy (due north) and Oz (vertically upwards), the coordinates of the points are as follows.

A: (0, 0, −15) B: (100, 0, −30)   C: (0, 100, −25)

D: (0, 0, −40) E: (100, 0, −50)   F: (0, 100, −35)

(i)   Verify that the Cartesian equation of the plane ABC is 3x + 2y + 20z + 300 = 0.

Ans:

If $3x + 2y + 20z + 300 = 0$ is the equation of the plane ABC, then points A, B, and C must satisfy that equation.

$Plane\ at\ A:\ 3 \times 0 + 2 \times 0 + 20 \times (-15) + 300 = -300 + 300 = 0 \Rightarrow OK$

$Plane\ at\ B:\ 3 \times 100 + 2 \times 0 + 20 \times (-30) + 300 = 300 - 600 + 300 = 0 \Rightarrow OK$

$Plane\ at\ C:\ 3 \times 0 + 2 \times 100 + 20 \times (-25) + 300 = 200 - 500 + 300 = 0 \Rightarrow OK$

OR

$let\ \overrightarrow{n}$ be a normal vector $\overrightarrow{n} = \begin{pmatrix} a \\ b \\ b \end{pmatrix}$, then the dot product of this normal and the

vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ should be zero.

$$\vec{AB} = \begin{pmatrix} 100 \\ 0 \\ -30 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -15 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ -15 \end{pmatrix}$$

$$\vec{AC} = \begin{pmatrix} 0 \\ 100 \\ -25 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -15 \end{pmatrix} = \begin{pmatrix} 0 \\ 100 \\ -10 \end{pmatrix}$$

$$\vec{n} \bullet \vec{AB} = \begin{pmatrix} a \\ b \\ b \end{pmatrix} \bullet \begin{pmatrix} 100 \\ 0 \\ -15 \end{pmatrix} = 100a + 0 - 15c = 0 \Rightarrow 20a = 3c$$

$$\vec{n} \bullet \vec{AC} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 100 \\ -10 \end{pmatrix} = 0 + 100b - 10c = 0 \Rightarrow 10b = c$$

$$20a = 3c \Rightarrow a = \frac{3c}{20} \Rightarrow a = 3 \; if \; c = 20 \Rightarrow b = \frac{c}{10} = \frac{20}{10} = 2$$

$$\vec{n} = \begin{pmatrix} a \\ b \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 20 \end{pmatrix} \Rightarrow Plane: \; 3x + 2y + 20z + d = 0 \Rightarrow at \; A(0,0,-15) \Rightarrow 0 + 20(-15) + d = 0$$

$$\Rightarrow d = 300$$

$$\Rightarrow Plane: \; 3x + 2y + 20z + 300 = 0$$

(ii) Find the vectors $\vec{DE}$ and $\vec{DF}$. Show that the vector $2i - j + 20k$ is perpendicular to each of these vectors. Hence find the Cartesian equation of the plane DEF.

Ans:

$$\vec{DE} = \begin{pmatrix} 100 \\ 0 \\ -50 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -40 \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ -10 \end{pmatrix}$$

$$\vec{DF} = \begin{pmatrix} 0 \\ 100 \\ -35 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -40 \end{pmatrix} = \begin{pmatrix} 0 \\ 100 \\ 5 \end{pmatrix}$$

$$\vec{n} \bullet \vec{DE} = \begin{pmatrix} 2 \\ -1 \\ 20 \end{pmatrix} \bullet \begin{pmatrix} 100 \\ 0 \\ -10 \end{pmatrix} = 200 + 0 - 200 = 0 \Rightarrow Ok$$

$$\vec{n} \bullet \vec{DF} = \begin{pmatrix} 2 \\ -1 \\ 20 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 100 \\ 5 \end{pmatrix} = -100 + 100 = 0 \Rightarrow Ok$$

$$\vec{n} = \begin{pmatrix} a \\ b \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 20 \end{pmatrix} \Rightarrow Plane: \ 2x - y + 20z + d = 0 \Rightarrow at \ D(0,0,-40) \Rightarrow 0 + 20(-40) + d = 0$$

$$\Rightarrow d = 800$$

$$\Rightarrow Plane: \ 2x - y + 20z = 800$$

(iii) By calculating the angle between their normal vectors, find the angle between the planes ABC and DEF.

$$\cos\theta = \frac{\begin{pmatrix} 3 \\ 2 \\ 20 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ 20 \end{pmatrix}}{\sqrt{3^2 + 2^2 + 20^2} \times \sqrt{2^2 + (-1)^2 + 20^2}} = 0.9878 \Rightarrow \theta = 8.95^0$$

It is decided to drill down to the seam from a point R(15, 34, 0) in a line perpendicular to the upper surface of the seam. This line meets the plane ABC at the point S.

(iv) Write down a vector equation of the line RS. Calculate the coordinates of S.

Ans: $$RS: r = \begin{pmatrix} 15 \\ 34 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 20 \end{pmatrix} = \begin{pmatrix} 15 + 3\lambda \\ 34 + 2\lambda \\ 20\lambda \end{pmatrix}$$

$$\Rightarrow 3(15 + 3\lambda) + 2(34 + 2\lambda) + 20 \times 20\lambda + 300 = 0$$

$$\Rightarrow 45 + 9\lambda + 68 + 4\lambda + 400\lambda + 300 = 0$$

$$\Rightarrow 413 + 413\lambda = 0$$

$$\Rightarrow \lambda = -1$$

So **S** is (12, 32, -20)

Ex-34-9: A skydiver drops from a helicopter. Before she opens her parachute, her speed $v\,ms^{-1}$ after time t seconds is modelled by the differential equation:

$$\frac{dv}{dt} = 10e^{-\frac{1}{2}t}$$

When $t = 0$, $v = 0$.

(i) Find v in terms of t.

Ans:

$$v = \int 10e^{-\frac{1}{2}t}\,dt = -20e^{-\frac{1}{2}t} + c$$

$$v(0) = -20 + c = 0$$

$$\Rightarrow c = 20$$

$$v(t) = 20 - 20e^{-\frac{1}{2}t}$$

(ii) According to this model, what is the speed of the skydiver in the long term.

Ans:

$$v(t) = 20 - 20e^{-\frac{1}{2}t}$$

$$v(\infty) = 20 - 0 = 20$$

So long term speed is $20\ m/s$

She opens her parachute when her speed is $10ms^{-1}$. Her speed t seconds after this is $w\,ms^{-1}$, and is modelled by the differential equation

$$\frac{dw}{dt} = -\frac{1}{2}(w-4)(w+5)$$

(iii) Express $\dfrac{1}{(w-4)(w+5)}$ in partial fractions.

Ans:

$$\frac{1}{(w-4)(w+5)} = \frac{A}{w-4} + \frac{B}{w+5}$$

$$A = \frac{1}{(w+5)}\bigg|_{w=4} = \frac{1}{4+5} = \frac{1}{9}$$

$$B = \frac{1}{(w-4)}\bigg|_{w=-5} = \frac{1}{-5-4} = -\frac{1}{9}$$

$$\frac{1}{(w-4)(w+5)} = \frac{A}{w-4} + \frac{B}{w+5} = \frac{1}{9(w-4)} - \frac{1}{9(w+5)}$$

(iv) Using this result, show that $\dfrac{w-4}{w+5} = 0.4e^{-4.5t}$.

Ans:

$$\frac{dw}{dt} = -\frac{1}{2}(w-4)(w+5)$$

$$\frac{dw}{(w-4)(w+5)} = -\frac{1}{2}dt$$

$$\int \frac{dw}{(w-4)(w+5)} = -\frac{1}{2}\int dt$$

$$\int \left(\frac{1}{9(w-4)} - \frac{1}{9(w+5)2} \right) dw = -\frac{1}{2}\int dt$$

$$\int \frac{dw}{9(w-4)} - \int \frac{dw}{9(w+5)} = -\frac{1}{2}\int dt$$

$$\frac{1}{9}\ln(w-4) - \frac{1}{9}\ln(w+5) = -\frac{1}{2}t + c$$

$$\frac{1}{9}\ln\left(\frac{w-4}{w+5}\right) = -\frac{1}{2}t + c$$

When t=0, w=10

$$\frac{1}{9}\ln\left(\frac{10-4}{10+5}\right)=0+c$$

$$\Rightarrow c=\frac{1}{9}\ln\frac{2}{5}$$

$$\frac{1}{9}\ln\left(\frac{w-4}{w+5}\right)=-\frac{1}{2}t+\frac{1}{9}\ln\frac{2}{5}$$

$$\frac{1}{9}\ln\left(\frac{w-4}{w+5}\right)-\frac{1}{9}\ln\frac{2}{5}=-\frac{1}{2}t$$

$$\frac{1}{9}\ln\left(\frac{w-4}{w+5}\div\frac{2}{5}\right)=-\frac{1}{2}t$$

$$\ln\left(\frac{5w-20}{2w+10}\right)=-\frac{9}{2}t=-4.5t$$

$$\frac{5w-20}{2w+10}=e^{-4.5t}$$

$$\frac{5(w-4)}{2(w+5)}=e^{-4.5t}$$

$$\Rightarrow\frac{w-4}{w+5}=\frac{2}{5}e^{-4.5t}=0.4e^{-4.5t}$$

(v) According to this model, what is the speed of the skydiver in the long term?

Ans:

PAPER-35

Section A(C4-9-6-10)

Ex-35-1: Express $\dfrac{x}{x^2-1}+\dfrac{2}{x+1}$ as a single fraction, simplifying your answer.

Ans:

$$\frac{x}{x^2-1}+\frac{2}{x+1}=\frac{x}{(x+1)(x-1)}+\frac{2}{x+1}=\frac{x+2(x-1)}{(x+1)(x-1)}=\frac{x+2x-2}{(x+1)(x-1)}=\frac{3x-2}{x^2-1}$$

Ex-35-2: Fig-35-2 shows the curve $y = \sqrt{1+x^2}$.
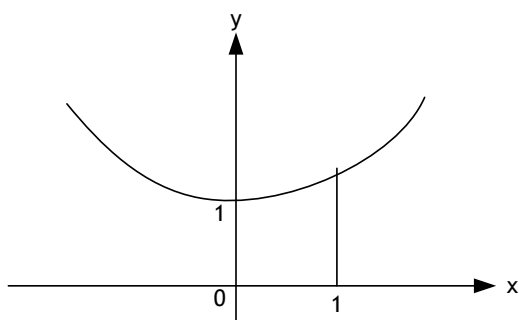
Fig-35-2

(i) The following table gives some values of x and y.

X	0	0.25	0.5	0.75	1
y	1	1.0308		1.25	1.4142

Find the missing value of y, giving your answer correct to 4 decimal places.

Ans:

$$y = \sqrt{1+x^2} = \sqrt{1+0.5^2} = \sqrt{1.25} = 1.1180$$

X	0	0.25	0.5	0.75	1
y	1	1.0308	**1.1180**	1.25	1.4142

Hence show that, using the trapezium rule with four strips, the shaded area is approximately 1.151 square units.

Ans:

$$Area \approx \frac{h}{2}\left(y_0 + y_4 + 2y_1 + 2y_2 + 2y_3\right)$$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

$$Area \approx \frac{0.25}{2}\left[1+1.4142 + 2(1.0308 + 1.1180 + 1.25)\right] = 0.125(9.2118) = 1.151475 \approx 1.151$$

Jenny uses a trapezium rule with 8 strips, and obtains a value of 1.158 square units. Explain why she must have made a mistake.

Ans:

Explain that the area is an over-estimate.

Or The curve is below the trapezia, so the area is an over-estimate.

This becomes less with more strips. **Or** greater number of strips improves accuracy so becomes less.

(ii)   The shaded area is rotated through 360^0 about the x-axis. Find the exact volume of the solid of revolution formed.

$$Volume = \int_0^1 \pi r^2 dx = \int_0^1 \pi y^2 dx = \int_0^1 \pi \left(\sqrt{1+x^2}\right)^2 dx = \pi \int_0^1 \left(1+x^2\right) dx = \pi \int_0^1 dx + \pi \int_0^1 x^2 dx$$

$$Volume = \pi \int_0^1 dx + \pi \int_0^1 x^2 dx = \pi \left[x - \frac{x^3}{3} \right]_0^1 = \pi \left(1 - \frac{1}{3}\right) = \frac{2\pi}{3}$$

Ex-35-3:   The parametric equations of a curve are

$$x = \cos 2\theta \qquad\qquad y = \sin\theta\cos\theta \qquad\qquad \text{for } 0 \le \theta \le \pi$$

Show that the Cartesian equation of the curve is $x^2 + 4y^2 = 1$

Sketch the curve.

Ans:

$$x = \cos 2\theta \qquad \Rightarrow x^2 = \cos^2 2\theta$$

$$y = \sin\theta\cos\theta = \frac{2\sin\theta\cos\theta}{2} = \frac{\sin 2\theta}{2} \qquad \Rightarrow y^2 = \frac{1}{4}\sin^2 2\theta \quad \Rightarrow \sin^2 2\theta = 4y^2$$

$$\Rightarrow \sin^2 2\theta + \cos^2 2\theta = 4y^2 + x^2 = 1 \qquad \Rightarrow x^2 + 4y^2 = 1$$
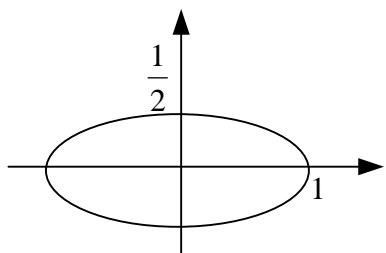
Ex-35-4:   Find the first three terms in the binomial expansion of $\sqrt{4+x}$ in ascending power of x.

State the set of values of x for which the expansion is valid.

Ans: This is from the Formula sheet. Use the general case.

Note: Binomial expansions

When n is a positive integer

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \ldots + b^n$$

Where

$$^nC_r = \frac{n!}{(n-r)!r!} \qquad\qquad \binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

General Case:

$$(a+b)^n = a^n + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \ldots$$

$$\sqrt{4+x} = (4+x)^{\frac{1}{2}} = 4^{\frac{1}{2}} + \frac{\frac{1}{2}\times 4^{\frac{1}{2}-1}x}{1!} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)4^{\frac{1}{2}-2}x^2}{2!} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)4^{\frac{1}{2}-3}x^3}{3!} + \ldots$$

$$\sqrt{4+x} = (4+x)^{\frac{1}{2}} = 2 + \frac{1}{4}x - \frac{x^2}{4\times 8\times 2!} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)4^{\frac{1}{2}-3}x^3}{3!} + \ldots$$

$$\sqrt{4+x} = (4+x)^{\frac{1}{2}} = 2 + \frac{1}{4}x - \frac{x^2}{64} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)4^{\frac{1}{2}-3}x^3}{3!} + \ldots$$

Valid for
$$-1 < \frac{x}{4} < 1$$
$$-4 < x < 4$$

Ex-35-5: **(i)** Express $\dfrac{3}{(y-2)(y+1)}$ in partial fractions.

(ii) Hence, given that x and y satisfy the differential equation

$$\frac{dy}{dx} = x^2(y-2)(y+1),$$

Show that $\dfrac{y-2}{y+1} = Ae^{x^3}$, where A is a constant.

Ans:

$$\frac{3}{(y-2)(y+1)} = \frac{A}{y-2} + \frac{B}{y+1} = \frac{A(y+1) + B(y-2)}{(y-2)(y+1)}$$

$$A(y+1) + B(y-2) = 3$$

$At \quad y = 2: \quad A(2+1) + 0 = 3 \Rightarrow 3A = 3 \Rightarrow A = 1$

$At \quad y = -1: \quad 0 - 3B = 3 \Rightarrow B = -1$

$$\Rightarrow \frac{3}{(y-2)(y+1)} = \frac{1}{y-2} - \frac{1}{y+1}$$

(ii) $\quad \dfrac{dy}{dx} = x^2(y-2)(y+1) \Rightarrow \dfrac{dy}{(y-2)(y+1)} = x^2 dx \Rightarrow \dfrac{3dy}{(y-2)(y+1)} = 3x^2 dx$

$$\Rightarrow \int \frac{3dy}{(y-2)(y+1)} = \int 3x^2 dx \quad \Rightarrow \int \frac{dy}{y-2} - \int \frac{dy}{y+1} = \int 3x^2 dx$$

$$\Rightarrow \ln(y-2) - \ln(y+1) = x^3 + c$$

$$\Rightarrow \ln\left(\frac{y-2}{y+1}\right) = x^3 + c \quad \Rightarrow \frac{y-2}{y+1} = e^{x^3+c} = e^{x^3} e^c = Ae^{x^3}$$

$$\Rightarrow \frac{y-2}{y+1} = Ae^{x^3}$$

Ex-35-6: Solve the equation $\tan(\theta + 45^0) = 1 - 2\tan\theta$, for $0^0 \le \theta \le \pi$

Ans: $\quad \tan(\theta + \alpha) = \dfrac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha}$

$$\tan(\theta + 45^0) = \frac{\tan\theta + \tan 45^0}{1 - \tan\theta\tan 45^0} = \frac{\tan\theta + 1}{1 - \tan\theta}$$

$$\Rightarrow \tan(\theta + 45^0) = \frac{\tan\theta + 1}{1 - \tan\theta} = 1 - 2\tan\theta \quad \Rightarrow (1 - \tan\theta)(1 - 2\tan\theta) = \tan\theta + 1$$

$$1 - 2\tan\theta - \tan\theta + 2\tan^2\theta = \tan\theta + 1 \quad \Rightarrow 2\tan^2\theta - 4\tan\theta = 0$$

$$\Rightarrow 2\tan\theta(\tan\theta - 2) = 0 \quad If \quad 2\tan\theta = 0 \quad \Rightarrow \theta = 0^0$$

$If \quad \tan\theta - 2 = 0 \quad \Rightarrow \tan\theta = 2 \quad \Rightarrow \theta = 63.435^0$

Ex-35-7: A straight pipeline AB passes through a mountain. With respect to axes $Oxyz$, with Ox due East, Oy due North and Oz vertically upwards, A has coordinates $(-200, 100, 0)$ and B has coordinates $(100, 200, 100)$, where units are metres.

(i) Verify that $\vec{AB} = \begin{pmatrix} 300 \\ 100 \\ 100 \end{pmatrix}$ and find the length of the pipeline.

Ans:

$$\vec{A} = \begin{pmatrix} -200 \\ 100 \\ 0 \end{pmatrix} \qquad\qquad \vec{B} = \begin{pmatrix} 100 \\ 200 \\ 100 \end{pmatrix}$$

$$\vec{AB} = \begin{pmatrix} 100 \\ 200 \\ 100 \end{pmatrix} - \begin{pmatrix} -200 \\ 100 \\ 0 \end{pmatrix} = \begin{pmatrix} 300 \\ 100 \\ 100 \end{pmatrix}$$

$$\left| AB \right| = \sqrt{(300)^2 + (100)^2 + (100)^2} = \sqrt{110000} = 331.662 \cong 332 \, m$$

(ii) Write down a vector equation of the line AB, and calculate the angle it makes with the vertical.

$$r = \begin{pmatrix} -200 \\ 100 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 300 \\ 100 \\ 100 \end{pmatrix}$$

Angle is between $r = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ **and** $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \cos\theta = \frac{3\times0 + 1\times0 + 1\times1}{\sqrt{11}\,\sqrt{1}} = \frac{1}{\sqrt{11}}$$

$$\Rightarrow \theta = 72.45^0$$

A thin flat layer of hard rock runs through the mountain. The equation of the plane containing this layer is $x + 2y + 3z = 320$

(iii) Find the coordinates of the point where the pipeline meets the layer of rock.

Ans: Meets plane of layer when

$$(-200+300\lambda)+2(100+100\lambda)3(100\lambda)=320$$

$$800\lambda = 320$$

$$\Rightarrow \lambda = \frac{2}{5}$$

$$r = \begin{pmatrix} -200 \\ 100 \\ 0 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} 300 \\ 100 \\ 100 \end{pmatrix} = \begin{pmatrix} -80 \\ 140 \\ 40 \end{pmatrix}$$

So meets layer at (-80, 140, 40)

(iv) By calculating the angle between the line AB and the normal to the plane of the layer, find the angle at which the pipeline cuts through the layer.

(iv) Normal to plane is $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Angle is between $r = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ **and** $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$\Rightarrow \cos\theta = \frac{3\times1+1\times2+1\times3}{\sqrt{11}\ \sqrt{14}} = \frac{8}{\sqrt{11}\sqrt{14}} = 0.6446$$
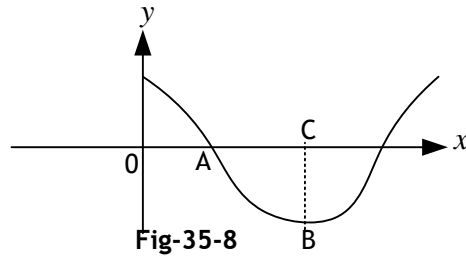
$$\Rightarrow \theta = 49.86^0$$

Angle with layer $= = 40.1^0$

Ex-35-8: Part of the track of a roller-coaster is modelled by a curve with the parametric equations

$$x = 2\theta - \sin\theta, \qquad y = 4\cos\theta \quad \text{for } 0 \le \theta \le 2\pi$$

This is shown in Fig-35-8. B is a minimum point, and BC is vertical.

Fig-35-8

(i) Find the values of the parameter at A and B.

Hence show that the ratio of the lengths OA and AC is $(\pi-1):(\pi+1)$

Ans:

At A, $y = 0 \Rightarrow y = 4\cos\theta = 0 \Rightarrow \theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \Rightarrow \theta = \dfrac{\pi}{2}$

$x\Big|_{\theta = \frac{\pi}{2}} = 2\left(\dfrac{\pi}{2}\right) - \sin\left(\dfrac{\pi}{2}\right) \Rightarrow x = \pi - 1$

At B, $y = Minimum \Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi \Rightarrow B = 4\cos\pi = -4$

$x\Big|_{\theta = \pi} = 2\pi - \sin\pi \Rightarrow x = 2\pi$

$A(\pi-1,0)$ $B(2\pi - 4)$

$OA = \pi - 1$ $AC = 2\pi - (\pi - 1) = \pi + 1$

$Ratio: \ (\pi-1):(\pi+1)$

(ii) Find $\dfrac{dy}{dx}$ in terms of θ. Find the gradient of the track at A.

Ans:

$x = 2\theta - \sin\theta,$ $y = 4\cos\theta$

$\dfrac{dx}{d\theta} = 2 - \cos\theta,$ $\dfrac{dy}{d\theta} = -4\sin\theta$

$\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx}$ Remember that $\dfrac{d\theta}{dx} = \dfrac{1}{\dfrac{dx}{d\theta}} = \dfrac{1}{2 - \cos\theta}$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = -4\sin\theta\left(\frac{1}{2-\cos\theta}\right) = -\frac{4\sin\theta}{2-\cos\theta}$$

$$\Rightarrow \frac{dy}{dx}\bigg|_{\theta=\frac{\pi}{2}} = -\frac{4\sin\theta}{2-\cos\theta}\bigg|_{\theta=\frac{\pi}{2}} = -\frac{4\sin\left(\frac{\pi}{2}\right)}{2-\cos\left(\frac{\pi}{2}\right)} = -\frac{4}{2} = -2$$

(iii) Show that, when the gradient of the track is 1, θ satisfies the

equation

$$\cos\theta - 4\sin\theta = 2$$

Ans:

$$\Rightarrow \frac{dy}{dx} = -\frac{4\sin\theta}{2-\cos\theta} = 1 \Rightarrow 2-\cos\theta = -4\sin\theta \Rightarrow \cos\theta - 4\sin\theta = 2 \Rightarrow OK$$

(iv) Express $\cos\theta - 4\sin\theta$ in the form $R\cos(\theta+\alpha)$

Hence solve the equation $\cos\theta - 4\sin\theta = 2$ for $0 \le \theta \le 2\pi$

Ans:

$$\cos\theta - 4\sin\theta = R\cos(\theta+\alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

$$\cos\theta - 4\sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$

$$R\cos\alpha = 1 \qquad\qquad ...(1)$$

$$-R\sin\alpha = -4 \qquad\qquad ...(2)$$

$$R\sin\alpha = 4 \qquad\qquad ...(2)$$

Dividing Eq.2 and Eq.1 $\qquad \tan\alpha = 4 \qquad \Rightarrow \alpha = 76^0$

$$R\cos 76^0 = 1 \Rightarrow R = \frac{1}{\cos 76^0} = 4.13$$

$$\Rightarrow \cos\theta - 4\sin\theta = R\cos(\theta+\alpha) = 4.13\cos(\theta+76^0)$$

$$\cos\theta - 4\sin\theta = 4.13\cos(\theta+76) = 2 \qquad \Rightarrow \cos(\theta+76^0) = \frac{2}{4.13} = 0.4843$$

$$\Rightarrow \theta+76^0 = \cos^{-1}(0.4843) = 61^0 \qquad \Rightarrow \theta = 61-76 = -15^0$$

$$\Rightarrow \theta+76^0 = 360^0+61^0 = 421^0 \qquad \Rightarrow \theta = 421-76 = 345^0$$

PAPER-36

Ex-36-1: Find the first three terms in the binomial expansion of $\dfrac{1+2x}{(1-2x)^2}$ in ascending powers of. State the set of values of x for which the expansion is valid.

Ans: The following is from the Formula sheet. See use general case:

Note: General case:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 \dots,$$

$$\frac{1+2x}{(1-2x)^2} = (1+2x)(1-2x)^{-2} = (1+2x)\left[1 + (-2)(-2x) + \frac{(-2)(-2-1)}{2!}(-2x)^2 + \dots.\right]$$

$$= (1+2x)\left[1 + 4x + 12x^2 + \dots\right]$$

$$= 1 + 4x + 12x^2 + 2x + 8x^2 + 24x^3 + \dots$$

Ex-36-2: Show that $\cot 2\theta = \dfrac{1-\tan^2\theta}{2\tan\theta}$

Hence solve the equation $\cot 2\theta = 1 + \tan\theta$ for $0^0 < \theta < 360^0$

Ans:

Remember: $\tan(\theta + \alpha) = \dfrac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha}$ **from the formulae sheet**

$$\tan 2\theta = \tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\cot 2\theta = \frac{1}{\tan 2\theta} = \frac{1}{\dfrac{2\tan\theta}{1-\tan^2\theta}} = \frac{1-\tan^2\theta}{2\tan\theta}$$

Hence solve the equation $\cot 2\theta = 1 + \tan\theta$ for $0^0 < \theta < 360^0$

$$\cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta} = 1+\tan\theta \quad \Rightarrow 2\tan\theta + 2\tan^2\theta = 1-\tan^2\theta \quad \Rightarrow 3\tan^2\theta + 2\tan\theta - 1 = 0$$

$$3\tan^2\theta + 2\tan\theta - 1 = 0 \quad \Rightarrow (3\tan\theta - 1)(\tan\theta + 1) = 0$$

$$\text{If} \quad (3\tan\theta - 1) = 0 \quad \Rightarrow 3\tan\theta = 1 \quad \Rightarrow \tan\theta = \frac{1}{3} \quad \Rightarrow \theta = 18.43^0, \; 198.43^0$$

$$\text{If} \quad (\tan\theta + 1) = 0 \quad \Rightarrow \tan\theta = -1 \quad \Rightarrow \theta = 135^0, \; 315^0$$

Ex-36-3: A curve has parametric equations

$$x = e^{2t} \qquad y = \frac{2t}{1+t}$$

(i) Find the gradient of the curve at the point where $t = 0$.
(ii) Find y in terms of x.

Ans:

$$x = e^{2t}, \qquad y = \frac{2t}{1+t}$$

$$\frac{dx}{dt} = 2e^{2t}, \qquad \frac{dy}{dt} = \frac{(1+t)\times 2 - 2t}{(1+t)^2} = \frac{2+2t-2t}{(1+t)^2} = \frac{2}{(1+t)^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad \text{Remember that} \quad \frac{dt}{dx} = \frac{1}{\dfrac{dx}{dt}} = \frac{1}{2e^{2t}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{2}{(1+t)^2}\left(\frac{1}{2e^{2t}}\right) = \frac{1}{e^{2t}(1+t^2)}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t=0} = \left.\frac{1}{e^{2t}(1+t^2)}\right|_{t=0} = 1$$

$$x = e^{2t} \;\Rightarrow 2t = \ln x \;\Rightarrow t = \frac{\ln x}{2} \qquad\qquad y = \frac{2t}{1+t} = \frac{2\left(\dfrac{\ln x}{2}\right)}{1+\dfrac{\ln x}{2}} = \frac{2\ln x}{2+\ln x}$$

Ex-36-4: The points A, B and C have coordinate $(1, 3, -2), (-1, 2, -3) \, and \, (0, -8, 1)$ respectively.

(i) Find the vectors $\vec{AB}$ and $\vec{AC}$

Ans:

$$\vec{AB} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} \qquad \vec{AC} = \begin{pmatrix} 0 \\ -8 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -11 \\ 3 \end{pmatrix}$$

(ii) Show that the vector $2i - j - 3k$ is perpendicular to the plane

ABC. Hence find the equation of the plane ABC.

Note: if the normal vector $\vec{n} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$ is perpendicular to the plane ABC, then it must also

be perpendicular to vectors $\vec{AB} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$ and $\vec{AC} = \begin{pmatrix} -1 \\ -11 \\ 3 \end{pmatrix}$, because these two vectors are

in the plane ABC. Therefore, the dot product of $\vec{n}$ and $\vec{AB}$ should be zero, or the dot

product of $\vec{n}$ and $\vec{AC}$ should be zero.

$$\vec{n}.\vec{AB} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}.\begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} = -4 + 1 + 3 = 0 \qquad \text{-> OK}$$

$$\vec{n}.\vec{AC} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}.\begin{pmatrix} -1 \\ -11 \\ 3 \end{pmatrix} = -2 + 11 - 9 = 0 \qquad \text{-> OK}$$

Note: If the normal to the plane is $\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = n_1 i + n_2 j + n_3 k$, then the equation of the

plane is: $n_1 x + n_2 y + n_3 z + d = 0$, where $d = -\vec{a}.\vec{n}$ and $\vec{a}$ is the position vector of a point
on the plane

$$2x - y - 3z + d = 0 \Rightarrow -d \Big|_{at\ A(1,3,-2)} = 2(1) - 3 - 3(-2) = 5 \Rightarrow d = -5$$

The equation of the plane: $2x - y - 3z - 5 = 0$

OR $2x - y - 3z + d = 0 \Rightarrow -d\Big|_{at\ B(-1,2,-3)} = 2(-1) - 2 - 3(-3) = 5 \Rightarrow d = -5$

The equation of the plane: $2x - y - 3z - 5 = 0$

Ex-36-5: **(i)** Verify that the lines $r = \begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ and $r = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

meet at the point (1, 3, 2).

Ans:

If the two lines meet at the same point, then both of them should have the same values.

$-5 + 3\lambda = -1 + 2\mu \Rightarrow 3\lambda - 2\mu = -1 + 5 = 4 \Rightarrow 3\lambda - 2\mu = 4$ ....(1)

$3 + 0 = 4 - \mu \Rightarrow \mu = 4 - 3 = 1 \Rightarrow 3\lambda = 4 + 2\mu = 4 + 2 = 6 \Rightarrow \lambda = 2$

$\lambda = 2$ **and** $\mu = 1$

$$r = \begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 4 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -5+6 \\ 3+0 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

And $$r = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1+2 \\ 4-1 \\ 2+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

Therefore, both of the lines satisfy the same point.

(ii) Find the acute angle between the lines.

Ans:

Angle between $\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ is θ, where

$$\cos\theta = \frac{\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 3 \times 2 + 0 \times (-1) + (-1) \times 0}{\sqrt{9+1} \times \sqrt{4+1}} = \frac{6}{\sqrt{50}} = \frac{6}{7.07} = 0.848 \quad \Rightarrow \theta = 31.95^0$$

PAPER-37

Some extra information:

Ex-37-1: Find the vector equation of the line passing through through the points (1, 3) and (5,8).

The direction of the line is $\begin{pmatrix} 5 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

The vector equation of the line is $r = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \end{pmatrix} \Rightarrow \frac{x-1}{4} = \frac{y-3}{5}$

Find the point of intersection of the lines $r = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $r = \begin{pmatrix} 6 \\ 1 \end{pmatrix} + k \begin{pmatrix} 3 \\ 8 \end{pmatrix}$

The same point satisfies both equations:

$1 + 4\lambda = 6 + 3k \Rightarrow 4\lambda - 3k = 5$ ....(1)

$3 + 3\lambda = 1 + 8k \Rightarrow 3\lambda - 8k = -2$ ....(2)

Solving the above equations: $\lambda = 2$ and $k = 1$, therefore point of intersection is

$$r = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+8 \\ 3+6 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \end{pmatrix}$$

And $r = \begin{pmatrix} 6 \\ 1 \end{pmatrix} + k \begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 6+3 \\ 1+8 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \end{pmatrix}$

Ex-37-2: Find the angle between the lines:

$$r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \quad \textbf{and} \quad r = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + k \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}}{\sqrt{2^2 + 0^2 + (-1)^2}\ \sqrt{1^2 + 1^2 + 3^2}} = \frac{2-3}{\sqrt{5}\ \sqrt{11}} = \frac{-1}{\sqrt{55}}$$

$$\theta = \cos^{-1}\left(-\frac{1}{\sqrt{55}}\right) = 97.7^0$$

Ex-37-3: The equation of a straight line through (a, b, c) with direction $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$ is given

by $r = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} d \\ e \\ f \end{pmatrix}$

OR $\dfrac{x-a}{d} = \dfrac{y-b}{e} = \dfrac{z-c}{f}$

Ex-37-4: Intersection of a plane and a line: Find the parametric form of the line and substitute into the plane.

E.g: $r = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a + \lambda d \\ b + \lambda e \\ c + \lambda f \end{pmatrix}$ into $n_1 x + n_2 y + n_3 z + d = 0$ will give an

equation in λ which can be solved.

Like: Find the intersection of the line $\dfrac{x-5}{-5} = \dfrac{y+2}{3} = \dfrac{z-1}{1}$ and the plane

$-5x + 3y + z = 5$

The line is $r = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} \Rightarrow x = 5 - 5\lambda,\ y = -2 + 3\lambda,\ z = 1 + \lambda$ and substitute into the

plane: $\Rightarrow -5(5 - 5\lambda) + 3(-2 + 3\lambda) + (1 + \lambda) = 5$

$\Rightarrow \lambda = 1 \Rightarrow \text{int } er\sec tion \text{ is } (0,\ 1,\ 2)$

Ex-37-5: In Fig-37-5, OAB is a thin bent rod, with OA=a meters, $AB = b$ metres and angle $OAB = 120^0$. The bent rod lies in a vertical plane. OA makes an angle θ above the horizontal. The vertical height BD of B above O is h metres. The horizontal through A meets BD at C and the vertical through A meets OD at E.
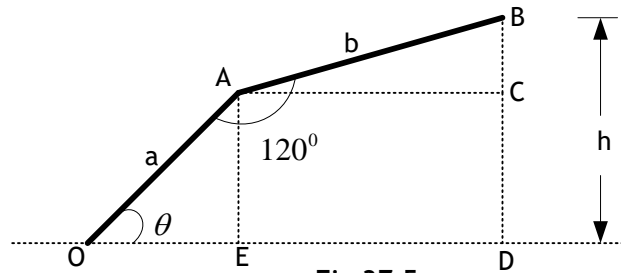

Fig-37-5

(i)　　Find angle BAC in terms of θ. Hence show that

$$h = a\sin\theta + b\sin(\theta - 60^0)$$

Ans:

$$Angle\ BAC = 120^0 - (90^0 - \theta) - 90^0 = \theta - 60^0$$

$$h = CD + BC \quad\quad CD = AE = a\sin\theta \quad\quad BC = b\sin(\theta - 60^0)$$

$$\Rightarrow h = CD + BC = a\sin\theta + b\sin(\theta - 60^0)$$

(ii)　　Hence show that $h = \left(a + \dfrac{1}{2}\right)\sin\theta - \dfrac{\sqrt{3}}{2}b\cos\theta$

$$h = a\sin\theta + b\sin(\theta - 60^0) = a\sin\theta + b\sin\theta\cos 60^0 - b\cos\theta\sin 60^0$$

$$h = a\sin\theta + \frac{b}{2}\sin\theta - \frac{b\sqrt{3}}{2}\cos\theta = \left(a + \frac{b}{2}\right)\sin\theta - \frac{\sqrt{3}}{2}b\cos\theta$$

The rod now rotates about O, so that θ varies. You may assume that the formulae for h in part (i) and (ii) remains valid.

(iii)　　Show that OB is horizontal when $\tan\theta = \dfrac{\sqrt{3}b}{2a + b}$

When OB is horizontal, then

$$h = 0 \quad \Rightarrow \left(a + \frac{b}{2} \right) \sin\theta - \frac{\sqrt{3}}{2} b\cos\theta = 0 \quad \Rightarrow \left(\frac{2a+b}{2} \right) \sin\theta = \frac{\sqrt{3}b}{2}\cos\theta$$

$$\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{\sqrt{3}b}{2a+b}$$

In the case when $a = 1$ and $b = 2$, $h = 2\sin\theta - \sqrt{3}\cos\theta$

$$h = \left(a + \frac{b}{2} \right) \sin\theta - \frac{\sqrt{3}}{2} b\cos\theta = \left(1 + \frac{2}{2} \right) \sin\theta - \frac{2\sqrt{3}}{2}\cos\theta = 2\sin\theta - \sqrt{3}\cos\theta$$

(iv) Express $2\sin\theta - \sqrt{3}\cos\theta$ in the form $R\sin(\theta - \alpha)$. Hence, for this case, write down the maximum value of h and the corresponding value of θ.

Ans:

$$2\sin\theta - \sqrt{3}\cos\theta = R\sin(\theta - \alpha) = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

$$2\sin\theta - \sqrt{3}\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

$$R\cos\alpha = 2 \qquad\qquad ...(1)$$

$$-R\sin\alpha = -\sqrt{3} \qquad\qquad ...(2)$$

$$R\sin\alpha = \sqrt{3} \qquad\qquad ...(2)$$

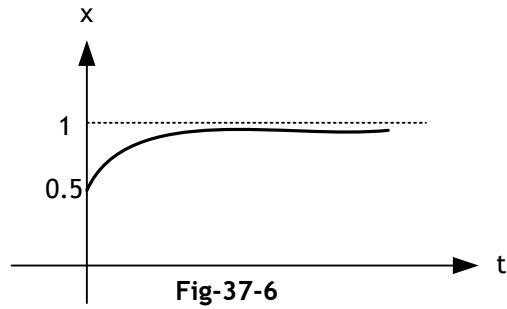Dividing Eq.2 and Eq.1 $\qquad \tan\alpha = \dfrac{\sqrt{3}}{2} \qquad \Rightarrow \alpha = 40.89^0$

$$R\cos 40.89^0 = 2 \quad \Rightarrow R = \frac{2}{\cos 40.89^0} = 2.646$$

$$\Rightarrow 2\sin\theta - \sqrt{3}\cos\theta = R\sin(\theta - \alpha) = 2.646\sin(\theta - 40.89^0)$$

$$h = 2\sin\theta - \sqrt{3}\cos\theta = 2.646\sin(\theta - 40.89^0)$$

$$h\Big|_{\max} \ when \ \theta - 40.89^0 = 90^0 \Rightarrow \theta = 90^0 + 40.89^0 = 130.89^0$$

Ex-37-6: Fig-37-6 illustrates the growth of a population with time. The proportion of the ultimate (long term) population is denoted by x, and the time in years by t. When $t = 0$, $x = 0.5$, and as t increases, x approaches 1.

Fig-37-6

One model for this situation is given by the differential equation

$$\frac{dx}{dt} = x(1-x)$$

(i)   Verify that $x = \dfrac{1}{1+e^{-t}}$ satisfies this differential equation, including the initial condition.

Ans:

$$x = \frac{1}{1+e^{-t}} = \left(1+e^{-t}\right)^{-1} \quad \Rightarrow \frac{dx}{dt} = -(-e^{-t})\left(1+e^{-t}\right)^{-2} = \frac{e^{-t}}{\left(1+e^{-t}\right)^2} = \left(\frac{1}{1+e^{-t}}\right)\left(\frac{e^{-t}}{1+e^{-t}}\right)$$

$$\Rightarrow \frac{dx}{dt} = \left(\frac{1}{1+e^{-t}}\right)\left(\frac{e^{-t}}{1+e^{-t}}\right) = \left(\frac{1}{1+e^{-t}}\right)\left(1-\frac{1}{1+e^{-t}}\right) = x(1-x)$$

(ii)   Find how long it will take, according to this model, for the population to reach three-quarters of its ultimate value.

$$x = \frac{1}{1+e^{-t}} \quad \Rightarrow ultimate \ value \ when \ t \to \infty \ \ ultimate \ value \ of \ x = 1$$

$$\frac{3}{4} = \frac{1}{1+e^{-t}} \quad \Rightarrow 3+3e^{-t} = 4 \quad \Rightarrow 3e^{-t} = 4-3 = 1 \quad \Rightarrow e^{-t} = \frac{1}{3} \quad \Rightarrow -t = \ln\left(\frac{1}{3}\right) \quad \Rightarrow t = -\ln\left(\frac{1}{3}\right)$$

An alternative model for this situation is given by the differential equation

$$\frac{dx}{dt} = x^2(1-x)$$

With $x = 0.5 \ when \ t = 0$ as before.

(iii)   Find constants A, B and C such that $\dfrac{1}{x^2(1-x)} = \dfrac{A}{x^2} + \dfrac{B}{x} + \dfrac{C}{1-x}$

$$\frac{1}{x^2(1-x)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{1-x} = \frac{A(1-x) + Bx(1-x) + Cx^2}{x^2(1-x)} \Rightarrow A(1-x) + Bx(1-x) + Cx^2 = 1$$

$$\Rightarrow A(1-x) + Bx(1-x) + Cx^2 = 1$$

$at\ x = 1:\quad 0 + 0 + C = 1 \Rightarrow C = 1$

$at\ x = 0:\quad A + 0 + 0 = 1 \Rightarrow A = 1$

$at\ x = 2:\quad -A - 2B + 4C = 1 \Rightarrow -1 - 2B + 4 = 1 \Rightarrow 2B = 2 \Rightarrow B = 1$

$A = 1\quad B = 1\quad C = 1$

(iv) Hence show that $t = 2 + \ln\left(\dfrac{x}{1-x}\right) - \dfrac{1}{x}$

$$\frac{dx}{dt} = x^2(1-x) \Rightarrow dt = \frac{dx}{x^2(1-x)} \Rightarrow \int dt = t = \int \frac{dx}{x^2(1-x)} = \int \frac{dx}{x^2} + \int \frac{dx}{x} + \int \frac{dx}{1-x} = -\frac{1}{x} + \ln x - \ln(1-x) + c$$

$$t = -\frac{1}{x} + \ln x - \ln(1-x) + c \Rightarrow 0 = -\frac{1}{0.5} + \ln(0.5) - \ln(1-0.5) + c \Rightarrow c = 2 - 0.693 + 0.693 = 2$$

$$\Rightarrow t = -\frac{1}{x} + \ln x - \ln(1-x) + 2 = 2 + \ln\left(\frac{x}{1-x}\right) - \frac{1}{x}$$

(v) Find how long it will take, according to this model, for the population to reach three-quarters of its ultimate value.

From the graph the ultimate value of the population (x) is 1.

$$t = 2 + \ln\left(\frac{x}{1-x}\right) - \frac{1}{x}$$

$$t = 2 + \ln\left(\frac{\frac{3}{4}}{1-\frac{3}{4}}\right) - \frac{1}{\frac{3}{4}} = \ln 3 + \frac{2}{3} = 1.77\ years$$
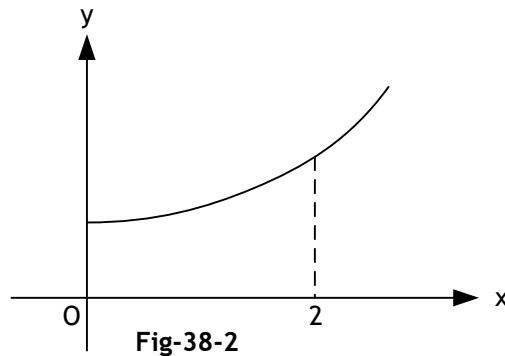
PAPER-38

Ex-38-1: Solve the equation $\dfrac{1}{x} + \dfrac{x}{x+2} = 1$

Ans:

$$\dfrac{1}{x} + \dfrac{x}{x+2} = 1 \quad \Rightarrow \dfrac{x+2+x^2}{x(x+2)} = 1 \quad \Rightarrow x^2 + x + 2 = x^2 + 2x \quad \Rightarrow 2x - x = x = 2$$

$$\Rightarrow x = 2$$

Ex-38-2: Fig-38-2 shows part of the curve $y = \sqrt{1+x^3}$

Fig-38-2

(i) Use the trapezium rule with 4 strips to estimate giving your answer correct to 3 significant figures.

Ans:

$$Area \approx \dfrac{h}{2}\left(y_0 + y_4 + 2y_1 + 2y_2 + 2y_3\right)$$

$$h = \dfrac{b-a}{n} = \dfrac{2-0}{4} = 0.5$$

$$y = \sqrt{1+x^3}$$

x	0	0.5	1	1.5	2
y	1	1.061	1.4142	2.092	3

$$Area \approx \dfrac{0.5}{2}\left[1+3+2(1.061+1.4142+2.092)\right] = 3.2836 \cong 3.28$$

(ii) Abdullah and Abdul Rahman each estimate the value of this integral using the trapezium rule with 8 strips.
Abdullah gets a result of 3.25, and Abdul Rahman gets 3.30. One of these results is correct. Without performing the calculation, state with a reason which is correct.

Ans:

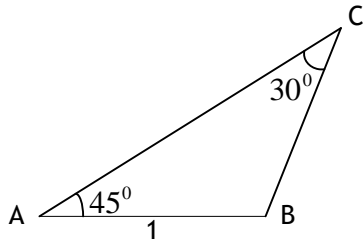Abdullah's value of 3.25 is correct. The area should decrese as the strips increases.

Ex-38-3: **(i)** Use the formula for $\sin(\theta + \phi)$, with $\theta = 45^0$ and $\phi = 60^0$ to show

that $\sin 105^0 = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

Ans:

$\sin(\theta + \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$

$\sin(45^0 + 60^0) = \sin(105^0) = \sin 45^0 \cos 60^0 + \cos 45^0 \sin 60^0$

$\Rightarrow \sin(105^0) = \dfrac{1}{\sqrt{2}} \times \dfrac{1}{2} + \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{3}}{2} = \dfrac{1+\sqrt{3}}{2\sqrt{2}}$

(ii) In triangle ABC, angle BAC = 45°, angle ACB = 30° and AB = 1 unit (**see Fig-38-3**).

Fig-38-3

Using the sine rule, together with the result in part (**i**), show that

$AC = \dfrac{\sqrt{3}+1}{\sqrt{2}}$

Ans:

$\dfrac{AC}{\sin B} = \dfrac{AB}{\sin C} \Rightarrow \dfrac{AC}{\sin(105^0)} = \dfrac{1}{\sin 30^0} \Rightarrow AC = \dfrac{\sin 105^0}{\sin 30^0} = \dfrac{\sqrt{3}+1}{(0.5)2\sqrt{2}} = \dfrac{\sqrt{3}+1}{\sqrt{2}}$

Ex-38-4: Show that $\dfrac{1 + \tan^2 \theta}{1 - \tan^2 \theta} = \sec 2\theta$

Hence, or otherwise, solve the equation for $\dfrac{1 + \tan^2 \theta}{1 - \tan^2 \theta} = 2$ for $0 \le \theta \le 180^0$

Ans:

$$\frac{1+\tan^2\theta}{1-\tan^2\theta} = \frac{1+\dfrac{\sin^2\theta}{\cos^2\theta}}{1-\dfrac{\sin^2\theta}{\cos^2\theta}} = \frac{\dfrac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}{\dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}} = \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta-\sin^2\theta} = \frac{1}{\cos(\theta+\theta)} = \sec 2\theta$$

$$\frac{1+\tan^2\theta}{1-\tan^2\theta} = \sec 2\theta = 2 \Rightarrow \cos 2\theta = \frac{1}{2} \Rightarrow 2\theta = 60^0,\ 300^0 \Rightarrow \theta = 30^0,\ 150^0$$

Ex-38-5: Find the first four terms in the binomial expansion of $(1+3x)^{\frac{1}{3}}$
State the range of values of x for which the expansion is valid.

Ans:

General Case:

$$(a+b)^n = a^n + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \dots$$

$$(1+b)^n = 1 + \frac{nb}{1!} + \frac{n(n-1)b^2}{2!} + \frac{n(n-1)(n-2)b^3}{3!} + \frac{n(n-1)(n-2)(n-3)b^4}{4!} + \dots$$

$$(1+3x)^{\frac{1}{3}} = 1 + \frac{\left(\frac{1}{3}\right)(3x)}{1!} + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)(3x)^2}{2!} + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)(3x)^3}{3!} + \dots$$

$$(1+3x)^{\frac{1}{3}} = 1 + x - x^2 + \frac{5}{3}x^3 + \dots$$

Valid for
$$-1 < 3x < 1$$
$$-\frac{1}{3} < x < \frac{1}{3}$$

Ex-38-6: **(i)** Express in partial fractions $\dfrac{1}{(2x+1)(x+1)}$.

Ans:

$$\frac{1}{(2x+1)(x+1)} = \frac{A}{2x+1} + \frac{B}{x+1} = \frac{A(1+x)+B(2x+1)}{(2x+1)(x+1)}$$

$$A(1+x)+B(2x+1) = 1$$

$$At \quad x = -1 \quad 0 + B(-2+1) = 1 \Rightarrow B = -1$$

At $x = 0$ $A + B = 1 \Rightarrow A = 1 - B = 1 - (-1) = 2$

$$\Rightarrow \frac{1}{(2x+1)(x+1)} = \frac{2}{2x+1} - \frac{1}{x+1}$$

OR

$$\frac{1}{(2x+1)(x+1)} = \frac{1}{2\left(x+\dfrac{1}{2}\right)(x+1)} = \frac{A}{x+\dfrac{1}{2}} + \frac{B}{x+1}$$

$$A = \left.\frac{1}{2(x+1)}\right|x = -0.5 = \frac{1}{2(-0.5+1)} = \frac{1}{2(0.5)} = 1$$

$$B = \left.\frac{1}{2(x+0.5)}\right|x = -1 = \frac{1}{2(-1+0.5)} = \frac{1}{2(-0.5)} = -1$$

$$\Rightarrow \frac{1}{(2x+1)(x+1)} = \frac{1}{x+\dfrac{1}{2}} - \frac{1}{x+1}$$

(ii) A curve passes through the point (0, 2) and satisfies the differential

equation $\dfrac{dy}{dx} = \dfrac{y}{(2x+1)(x+1)}$

Show by integration that $y = \dfrac{4x+1}{x+1}$

$$\frac{dy}{dx} = \frac{y}{(2x+1)(x+1)} \quad \Rightarrow \frac{dy}{y} = \left(\frac{2}{2x+1} - \frac{1}{x+1}\right)dx$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{2dx}{2x+1} - \int \frac{dx}{x+1}$$

$$\Rightarrow \ln y = \ln(2x+1) - \ln(x+1) + c$$

$$\Rightarrow \ln y = \ln(2x+1) - \ln(x+1) + c = \ln\left(\frac{A(2x+1)}{x+1}\right) \Rightarrow y = \frac{A(2x+1)}{x+1}$$

$$\Rightarrow y = \frac{A(2x+1)}{x+1}$$

$$2 = \frac{A}{1} \Rightarrow A = 2$$

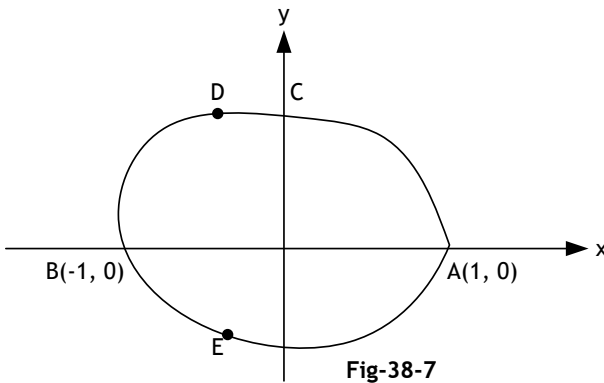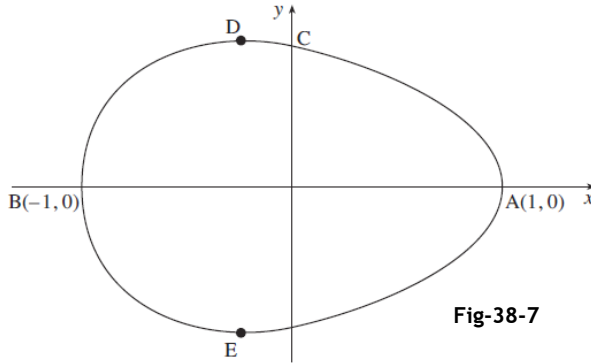$$\Rightarrow y = \frac{2(2x+1)}{x+1} = \frac{4x+2}{x+1}$$

Ex-38-7: Fig-38-7 shows the curve with parametric equations

$$x = \cos\theta, \qquad y = \sin\theta - \frac{1}{8}\sin 2\theta, \quad 0 \le \theta \le 2\pi$$

The curve crosses the x-axis at points A(1, 0) and B(-1,0), and the positive y-axis at C. D is the maximum point of the curve, and E is the minimum point.

The solid of revolution formed when this curve is rotated through 360° about the x-axis is used to model the shape of an egg.

Fig-38-7

Fig-38-7

(i) Show that, at the point A, $\theta = 0$. Write down the value of θ at the point B, and find the coordinates of C.

Ans:

at A, $\cos\theta = 1 \Rightarrow \theta = 0$

at B, $\cos\theta = -1 \Rightarrow \theta = \pi$

at C, $x = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \dfrac{\pi}{2}$

$\Rightarrow y = \sin\dfrac{\pi}{2} - \dfrac{1}{8}\sin\pi = 1$

(ii) Find $\dfrac{dy}{dx}$ in terms of θ.

Hence show that, at the point D,

$2\cos^2\theta - 4\cos\theta - 1 = 0$.

Ans:

$$\frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\cos\theta - \dfrac{1}{4}\cos 2\theta}{-\sin\theta} = \frac{\cos 2\theta - 4\cos\theta}{4\sin\theta}$$

$$\frac{dy}{dx} = 0 \text{ when } \cos 2\theta - 4\cos\theta = 0$$

$$\Rightarrow 2\cos^2\theta - 1 - 4\cos\theta = 0$$

(iii) Solve this equation, and hence find the *y*-coordinate of D, giving your answer correct to 2 decimal places.
Ans:

$$\cos\theta = \frac{4 \pm \sqrt{16+8}}{4} = 1 \pm \frac{1}{2}\sqrt{6}$$

$$\cos\theta = \begin{cases} 1 + \dfrac{1}{2}\sqrt{6} \text{ } no \text{ } good \\[3mm] 1 - \dfrac{1}{2}\sqrt{6} \text{ } ok \end{cases}$$

$$\Rightarrow \cos\theta = 1 - \frac{1}{2}\sqrt{6} \Rightarrow \theta = 1.7975 \text{ } rad$$

$$\Rightarrow y = \sin\theta - \frac{1}{8}\sin 2\theta = 1.0292$$

The Cartesian equation of the curve (for $0 \le \theta \le \pi$) is
$$y = \frac{1}{4}(4-x)\sqrt{1-x^2} \ .$$

(iv) Show that the volume of the solid of revolution of this curve about the *x*-axis is given by
$$\frac{1}{16}\pi \int_{-1}^{1} \left(16 - 8x - 15x^2 + 8x^3 - x^4\right)dx$$
Evaluate this integral.

Ans: $V = \int_{-1}^{1} \pi y^2 dx = \frac{1}{16}\pi \int_{-1}^{1} \left(16 - 8x + x^2\right)\left(1 - x^2\right)dx$

$$V = \frac{1}{16}\pi \int_{-1}^{1} \left(16 - 8x - 15x^2 + 8x^3 - x^4\right)dx$$

$$V = \frac{\pi}{16}\left[16x - \frac{8}{2}x^2 - \frac{15}{3}x^3 + \frac{8}{4}x^4 - \frac{x^5}{5}\right]\Bigg|_{-1}^{1}$$

$$V = \frac{\pi}{16}\left[32 - 10 - \frac{2}{5}\right] = 1.35\pi = 4.24$$

Ex-38-8: A pipeline is to be drilled under a river (see Fig-38-8). With respect to axes O*xyz*, with the *x*-axis pointing East, the *y*-axis North and the *z*-axis vertical, the pipeline is to consist of a straight section AB from the point to the point directly under the river, and another straight section BC. All lengths are in metres.

Fig-38-8
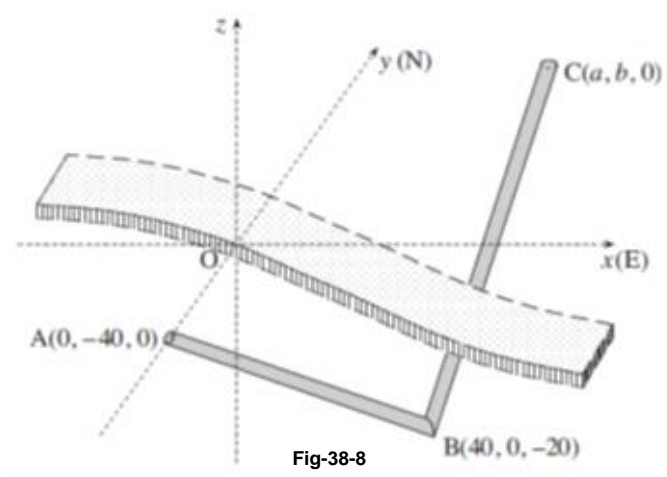
(i) Calculate the distance AB.
Ans:
$$AB = \sqrt{(40-0)^2 + (0+40)^2 + (-20-0)^2} = 60 \ m$$

The section BC is to be drilled in the direction of the vector .

(ii) Find the angle ABC between the sections AB and BC.
Ans:
$$\vec{BA} = \begin{pmatrix} -40 \\ -40 \\ 20 \end{pmatrix} = 20\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}}{\sqrt{9}\ \sqrt{26}} = -\frac{13}{3\sqrt{26}}$$

$$\Rightarrow \theta = 148^0$$

The section BC reaches ground level at the point C(a, b, 0).

(iii) Write down a vector equation of the line BC. Hence find *a* and *b*.
Ans:

$$r = \begin{pmatrix} 40 \\ 0 \\ -20 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

at C: $z = 0 \Rightarrow \lambda = 20$

$$\Rightarrow a = 40 + 3(20) = 100$$

$$b = 0 + 4(20) = 80$$

(iv) Show that the vector $6i - 5j + 2k$ is perpendicular to the plane ABC. Hence find the Cartesian equation of this plane.
Ans:

PAPER-39

SECTION A (C4-1-1-08)

Ex-39-1: Express $3\cos x + 4\sin\theta$ in the form $R\cos(x - \alpha)$, where $R > 0$

and $0 < \alpha < \dfrac{\pi}{2}$.

Hence solve the equation $3\cos x + 4\sin\theta = 2$ for $-\pi \le \theta \le \pi$.

Ans:

$$3\cos\theta + 4\sin\theta = R\cos(\theta - \alpha) = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

$$3\cos\theta + 4\sin\theta = R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$$

$$R\cos\alpha = 3 \qquad\qquad ...(1)$$

$$R\sin\alpha = 4 \qquad\qquad ...(2)$$

Dividing Eq.2 and Eq.1 $\qquad \tan\alpha = \dfrac{4}{3} \qquad \Rightarrow \alpha = 53^0$

$$R\cos 53^0 = 3 \Rightarrow R = \dfrac{3}{\cos 53^0} = \dfrac{3}{0.6} = 5$$

$$\Rightarrow 3\cos\theta + 4\sin\theta = R\cos(\theta - \alpha) = 5\cos(\theta - 53^0) = 5\cos(\theta - 53^0)$$

$$3\cos x + 4\sin\theta = 2 \quad \Rightarrow 5\cos(\theta - 53^0) = 2 \quad \Rightarrow \cos(\theta - 53^0) = \dfrac{2}{5} \quad \Rightarrow \theta - 53^0 = 66.42$$

$$\Rightarrow \theta = 66.42^0 + 53^0 = 119.42^0$$

OR $\quad \cos(\theta - 53^0) = \dfrac{2}{5} \quad \Rightarrow \theta - 53^0 = -66.42^0 \quad \Rightarrow \theta = -13.42^0$

Ex-39-2: **(i)** Find the first three terms in the binomial expansion of $\dfrac{1}{\sqrt{1 - 2x}}$

State the set of values of x for which the expansion is valid.

Ans:

$$(1 + zx)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + ... + \dfrac{n(n-1)...(n-r+1)}{1 \times 2 \times 3 \times 4 ... \times r}x^r$$

$$\dfrac{1}{\sqrt{1 - 2x}} = \dfrac{1}{(1 - 2x)^{\frac{1}{2}}} = (1 + (-2x))^{-\frac{1}{2}} = \left[1 + \left(-\dfrac{1}{2}\right)(-2x) + \dfrac{\left(-\dfrac{1}{2}\right)\left(-\dfrac{1}{2} - 1\right)}{2}(-2x)^2 + ...\right]$$

$$\Rightarrow \dfrac{1}{\sqrt{1 - 2x}} = \left[1 + \left(-\dfrac{1}{2}\right)(-2x) + \dfrac{\left(-\dfrac{1}{2}\right)\left(-\dfrac{1}{2} - 1\right)}{2}(-2x)^2 + ...\right] = \left[1 + x + \dfrac{3}{2}x^2 + ...\right]$$

Valid for $\left|-2x\right| < 1 \Rightarrow -1 < -2x < 1 \Rightarrow -\dfrac{1}{2} < x < \dfrac{1}{2}$ **from Formula sheet.**

(ii) Hence find the first three terms in the series expansion of $\dfrac{1 + 2x}{\sqrt{1 - 2x}}$.
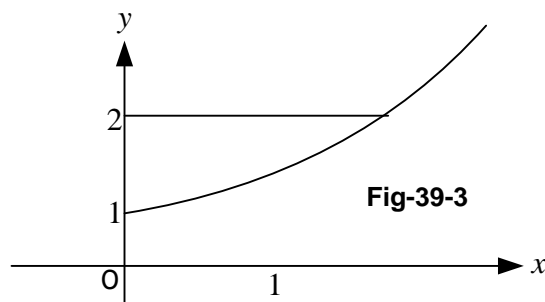
$$\frac{1+2x}{\sqrt{1-2x}} = \frac{1+2x}{(1-2x)^{\frac{1}{2}}} = (1+2x)(1+(-2x))^{-\frac{1}{2}} = (1+2x)\left[1+\left(-\frac{1}{2}\right)(-2x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2}(-2x)^2+...\right]$$

$$\Rightarrow \frac{1+2x}{\sqrt{1-2x}} = (1+2x)\left[1+\left(-\frac{1}{2}\right)(-2x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2}(-2x)^2+...\right] = (1+2x)\left[1+x+\frac{3}{2}x^2+....\right]$$

$$\Rightarrow \frac{1+2x}{\sqrt{1-2x}} = 1+x+\frac{3}{2}x^2+2x+2x^2+3x^3+...=1+3x+\frac{7}{2}x^2+3x^3+...$$
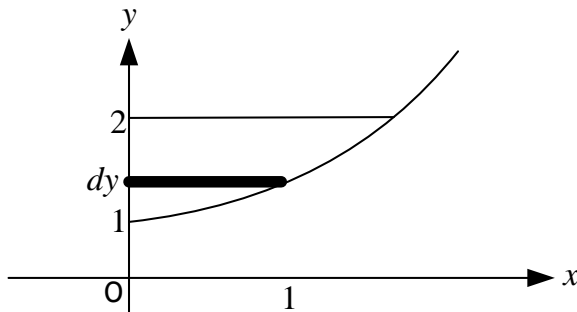
Ex-39-3:   Fig-39-3 shows part of the curve $y=1+x^2$, together with the line y = 2.

Fig-39-3

The region enclosed by the curve, the y-axis and the line y = 2 is rotated through 360^0 about the y-axis.

Find the volume of the solid generated, giving your answer in terms of π.

Ans:

$$V = \pi\int_{1}^{2}r^2dy = \pi\int_{1}^{2}x^2dy = \pi\int_{1}^{2}(y-1)dy = \pi\left(\frac{y^2}{2}-y\right)\Big|_{1}^{2} = \pi\left(\frac{4}{2}-2\right)-\pi\left(\frac{1}{2}-1\right) = \frac{\pi}{2}$$

Ex-39-4: The angle θ satisfies the equation $\sin\left(\theta + 45^0\right) = \cos\theta$.

(i) Using the exact values of $\sin 45^0$ and $\cos 45^0$, show that $\tan\theta = \sqrt{2} - 1$.

Ans: $\sin\left(\theta + 45^0\right) = \sin\theta\cos 45^0 + \cos\theta\sin 45^0 = \cos\theta$

$$\sin\left(\theta + 45^0\right) = \frac{\sqrt{2}}{2}\sin\theta + \frac{\sqrt{2}}{2}\cos\theta = \cos\theta$$

$$\Rightarrow \frac{\sqrt{2}}{2}\sin\theta = \cos\theta - \frac{\sqrt{2}}{2}\cos\theta = \left(1 - \frac{\sqrt{2}}{2}\right)\cos\theta = \left(\frac{2 - \sqrt{2}}{2}\right)\cos\theta$$

$$\Rightarrow \frac{\sqrt{2}}{2}\sin\theta = \left(\frac{2 - \sqrt{2}}{2}\right)\cos\theta \Rightarrow \frac{\sin\theta}{\cos\theta} = \tan\theta = \left(\frac{2 - \sqrt{2}}{2}\right)\left(\frac{2}{\sqrt{2}}\right) = \sqrt{2} - 1$$

(ii) Find the values of θ for $0^0 < 360^0$.

Ans: $\tan\theta = \sqrt{2} - 1 = 0.4142 \qquad \Rightarrow \theta = 22.5^0,\ \ 202.5^0$

Ex-39-5: Express $\dfrac{4}{x(x^2 + 4)}$ in partial fractions.

Ans:

$$\frac{4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

$$\frac{4}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4} = \frac{A(x^2 + 4) + (Bx + C)x}{x(x^2 + 4)}$$

$$\Rightarrow Ax^2 + 4A + Bx^2 + Cx = 4$$

at $x = 0 \quad \Rightarrow 0 + 4A + 0 + 0 = 4 \quad \Rightarrow 4A = 4 \Rightarrow A = 1$

at $x = 1 \Rightarrow A + 4A + B + C = 4 \Rightarrow 5A + B + C = 4 \Rightarrow B + C = 4 - 5 = -1 \qquad ...(1)$

at $x = 2 \quad \Rightarrow 4A + 4A + 4B + 2C = 4 \quad \Rightarrow 4B + 2C = 4 - 8A = -4 \qquad ...(2)$

Subtracting Eq.1 from 2 $\qquad B = -1$ and $C = 0$

$$\frac{4}{x(x^2 + 4)} = \frac{1}{x} - \frac{x}{x^2 + 4}$$

Ex-39-6:   Solve the equation $\cos ec\,\theta = 3$, for $0^0 < \theta < 360^{\,0}$.

Ans:

$\cos ec\,\theta = 3$

$\cos ec\,\theta = \dfrac{1}{\sin\,\theta} = 3 \quad \Rightarrow \sin\,\theta = \dfrac{1}{3} \quad \Rightarrow \theta = 19.47^0,\; 160.52^0$

SECTION B (P-39)

Ex-39-7:   A glass ornament OABCDEFG is a truncated pyramid on a rectangular base (see Fig-39- 7). All dimensions are in centimetres.
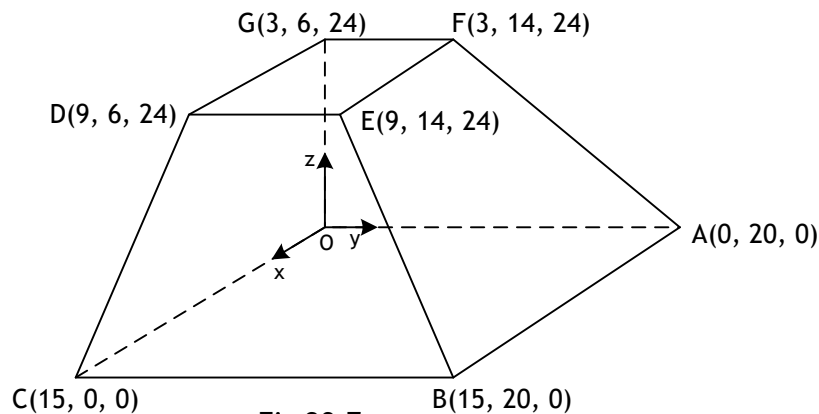

Fig-39-7

(i)   Write down the vectors $\overrightarrow{CD}$ and $\overrightarrow{CB}$.

Ans:

$$\overrightarrow{CD} = (9 - 15)i + (6 - 0)j + (24 - 0)k = -6i + 6j + 24k$$

$$\overrightarrow{CB} = (15 - 15)i + (20 - 0)j + (0 - 0)k = 0i + 20j + 0k = 20j$$

(ii)   Find the length of the edge CD.

Ans:

$$\overrightarrow{CD} = -6i + 6j + 24k$$

$$\left|CD\right| = \sqrt{6^2 + 6^2 + 24^2}$$

(iii) Show that the vector 4**i** + **k** is perpendicular to the vectors $\overrightarrow{CD}$ and $\overrightarrow{CB}$. Hence find the Cartesian equation of the plane BCDE.

If $n = 4i + k$ is perpendicular to $\overrightarrow{CD} = -6i + 6j + 24k$, then their dot product should be zero.

$$\vec{n} \bullet \overrightarrow{CD} = (4i + k) \bullet (-6i + 6j + 24k) = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -6 \\ 6 \\ 24 \end{pmatrix} = -24 + 24 = 0 \Rightarrow OK$$

$$\vec{n} \bullet \overrightarrow{CB} = (4i + k) \bullet (20j) = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 0 \\ 20 \\ 0 \end{pmatrix} = 0 \Rightarrow OK$$

Note: The plane through point $P_0(x_0, y_0, z_0)$ perpendicular to $\vec{n} = ai + bj + ck$ has

the equation: $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \bullet \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} x - 15 \\ y - 0 \\ z - 0 \end{pmatrix} = 0 \Rightarrow 4(x - 15) + z = 0 \Rightarrow 4x + z = 60$$

OR

$$\Rightarrow \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} x - 15 \\ y - 20 \\ z - 0 \end{pmatrix} = 0 \Rightarrow 4(x - 15) + z = 0 \Rightarrow 4x + z = 60$$

OR

$$\Rightarrow \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} x - 9 \\ y - 6 \\ z - 24 \end{pmatrix} = 0 \Rightarrow 4(x - 9) + z - 24 = 0 \Rightarrow 4x + z = 60$$

(iv) Write down vector equations for the lines OG and AF.

Show that they meet at the point P with coordinates (5, 10, 40). [5]

Ans:

$$\vec{OG} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 6 \\ 24 \end{pmatrix}$$

$$\vec{AF} = \begin{pmatrix} 0 \\ 20 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -6 \\ 24 \end{pmatrix}$$

Note: For finding the point of intersection of the two lines, the same point should satisfy both equations:

$$3\lambda = 3\mu \Rightarrow \lambda = \mu$$

$$6\lambda = 20 - 6\mu \Rightarrow 6\lambda + 6\mu = 12\lambda = 20 \Rightarrow \lambda = \frac{5}{3}$$

$$\Rightarrow \vec{OG} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 6 \\ 24 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \frac{5}{3}\begin{pmatrix} 3 \\ 6 \\ 24 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 40 \end{pmatrix}$$

$$\Rightarrow \vec{AF} = \begin{pmatrix} 0 \\ 20 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -6 \\ 24 \end{pmatrix} = \begin{pmatrix} 0 \\ 20 \\ 0 \end{pmatrix} + \frac{5}{3}\begin{pmatrix} 3 \\ -6 \\ 24 \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 40 \end{pmatrix}$$

Therefore they both meet at point P(5, 10, 40)

You may assume that the lines CD and BE also meet at the point P.

The volume of a pyramid is $\frac{1}{3} \times$ area of base $\times$ height.

(v) Find the volumes of the pyramids POABC and PDEFG.

Hence find the volume of the ornament.

Ans:

$$h = 40$$

$$POABC : V = \frac{1}{3} \times 20 \times 15 \times 40 = 4000 \ cm^3$$

$$PDEFG : V = \frac{1}{3} \times 8 \times 6 \times (40 - 24) = 256 \ cm^3$$

$$\Rightarrow vol \ of \ ornament = 4000 - 256 = 3744 \ cm^3$$

Ex-39-8:   A curve has equation

$$x^2 + 4y^2 = k^2$$

where k is a positive constant.

(i)   Verify that $x = k \cos \theta$, $y = \frac{1}{2} k \sin \theta$

are parametric equations for the curve.

Ans:

If $x = k \cos \theta$ and $y = \frac{1}{2} k \sin \theta$ are the parametric equation of the above equation, then it should satisfy that equation:

$$x = k \cos \theta \Rightarrow x^2 = k^2 \cos^2 \theta \Rightarrow \cos^2 \theta = \frac{x^2}{k^2}$$

$$y = \frac{1}{2} k \sin \theta \Rightarrow y^2 = \frac{k^2}{4} \sin^2 \theta \Rightarrow \sin^2 \theta = \frac{4y^2}{k^2}$$

$$\sin^2 \theta + \cos^2 \theta = 1 \Rightarrow \frac{4y^2}{k^2} + \frac{x^2}{k^2} = 1 \Rightarrow x^2 + 4y^2 = k^2$$

(ii)   Hence or otherwise show that $\dfrac{dy}{dx} = -\dfrac{x}{4y}$.
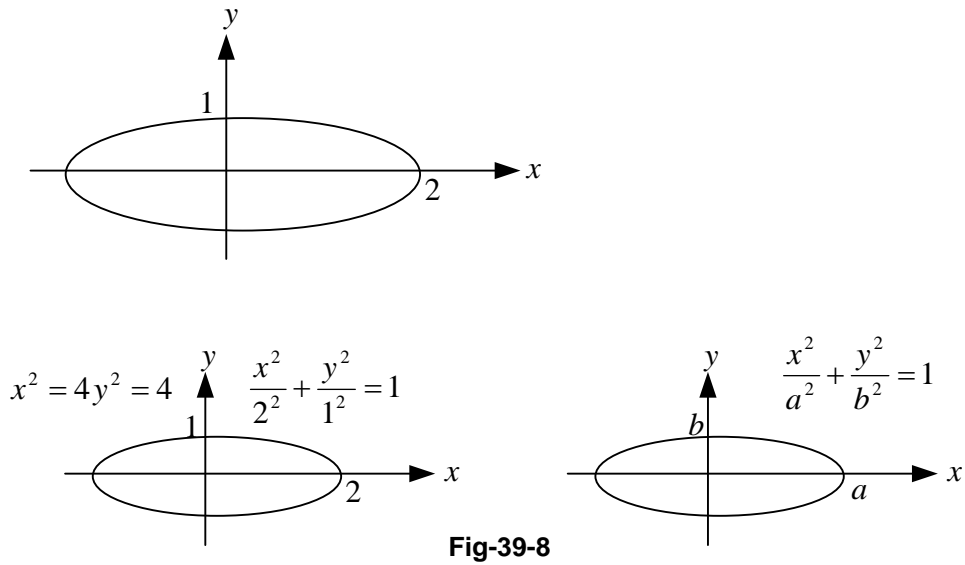
Ans:

$$x = k \cos \theta$$

$$\frac{dx}{d\theta} = -k \sin \theta$$

$$y = \frac{1}{2} k \sin \theta \quad \Rightarrow \frac{dy}{d\theta} = \frac{1}{2} k \cos \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \left(\frac{1}{2} k \cos \theta \right) \left(-\frac{1}{k \sin \theta} \right) = (k \cos \theta) \left(-\frac{1}{2k \sin \theta} \right) = -\frac{x}{4y}$$

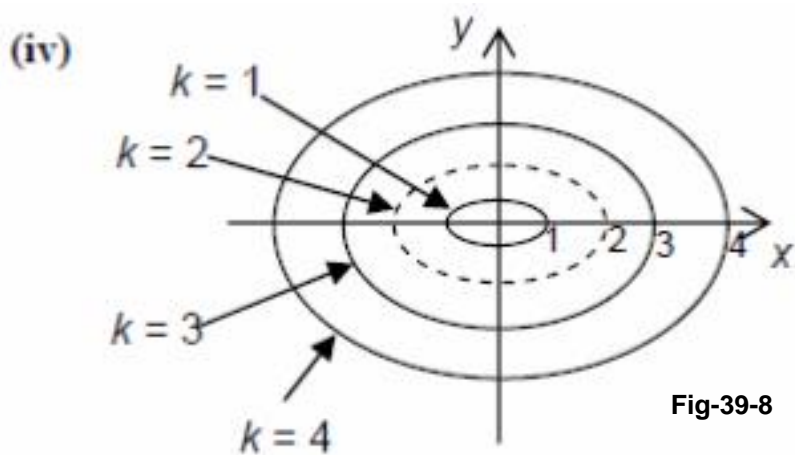**(iii)** Fig-39-8 illustrates the curve for a particular value of k. Write down this value of k

$x^2 = 4y^2 = 4$ $\dfrac{x^2}{2^2} + \dfrac{y^2}{1^2} = 1$ $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Fig-39-8

Ans:

$$x^2 + 4y^2 = k^2 \qquad \text{...(1)}$$

$$x^2 + 4y^2 = 4 \qquad \text{...(2)}$$

Comparing Eq.1 and 2, then $k^2 = 4 \Rightarrow k = 2$

(iv) Copy Fig. 8 and on the same axes sketch the curves for k = 1, k = 3 and k = 4.

Ans:

(iv) $k = 1$ $k = 2$ $k = 3$ $k = 4$

Fig-39-8

On a map, the curves represent the contours of a mountain. A stream flows down the mountain. Its path on the map is always at right angles to the contour it is crossing.

(v)   Explain why the path of the stream is modelled by the differential equation

$$\frac{dy}{dx} = \frac{4y}{x}.$$

Ans:

$$grad\ of\ stream\ path = -\frac{1}{grad\ of\ contour}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{\left(-\dfrac{x}{4y}\right)} = \frac{4y}{x}$$

(vi)   Solve this differential equation.

Ans:

$$\frac{dy}{dx} = \frac{4y}{x}$$

$$\frac{dy}{4y} = \frac{dx}{x}$$

$$\int \frac{dy}{4y} = \int \frac{dx}{x}$$

$$\frac{1}{4}\ln y = \ln x + \ln k$$

$$\ln y = 4\ln x + 4\ln k = \ln(kx)^4$$

$$y = k_1 x^4$$

$$1 = 16k_1 \quad \Rightarrow k_1 = \frac{1}{16}$$

$$y = \frac{x^4}{16}$$

Given that the path of the stream passes through the point (2, 1), show that its

equation is $y = \dfrac{x^4}{16}$.

PAPER-40

SECTION A (C4-1-6-07)

Ex-40-1: Express $\sin\theta - 3\cos\theta$ in the form $R\sin(\theta - \alpha)$, where R and α are constants to be determined, and $0^0 < \alpha < 90^0$.

Ans:

$$\sin\theta - 3\cos\theta = R\sin(\theta - \alpha) = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

$$\sin\theta - 3\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$

$$R\cos\alpha = 1 \qquad ...(1)$$

$$-R\sin\alpha = -3 \qquad ...(2)$$

$$R\sin\alpha = 3 \qquad ...(2)$$

Dividing Eq.2 and Eq.1 $\qquad \tan\alpha = 3 \qquad \Rightarrow \alpha = 71.57^0$
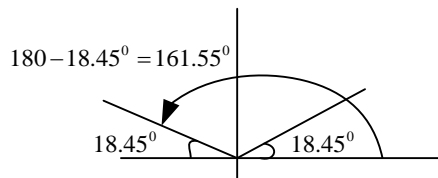
$$R\cos 71.57^0 = 1 \Rightarrow R = \frac{1}{\cos 71.57^0} = \frac{1}{0.316} = 3.16$$

$$\Rightarrow \sin\theta - 3\cos\theta = R\sin(\theta - \alpha) = 3.16\sin(\theta - 71.57^0) = 3.16\sin(\theta - 71.57^0)$$

Hence solve the equation $\sin\theta - 3\cos\theta = 1$ for $0^0 \le \theta \le 360^0$.

$$\sin\theta - 3\cos\theta = 1 \quad \Rightarrow 3.16\sin(\theta - 71.57^0) = 1 \quad \Rightarrow \sin(\theta - 71.57^0) = \frac{1}{3.16}$$

$$\Rightarrow \theta - 71.57^0 = 18.45^0$$

$$\Rightarrow \theta - 71.57^0 = 18.45^0 \Rightarrow \theta = 18.45^0 + 71.57^0 = 90.02^0$$

$$and \quad \theta - 71.57^0 = 161.55^0 \Rightarrow \theta = 161.55^0 + 71.57^0 = 233.12^0$$

Ex-40-2: Write down normal vectors to the plane $2x + 3y + 4z = 10$ and $x - 2y + z = 5$.
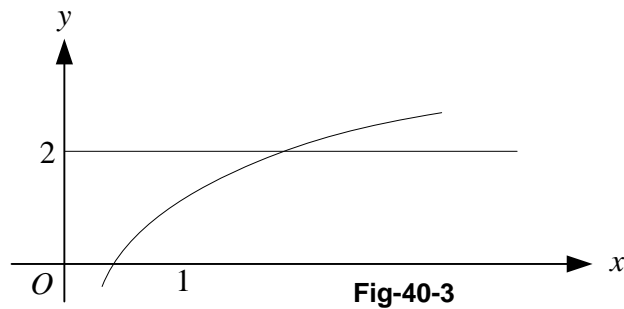
Hence show that these planes are perpendicular to each other.

Ans:

$$\vec{N_1} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \vec{N_2} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\vec{N_1} . \vec{N_2} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 2 \times 1 + 3 \times (-2) + 4 \times 1 = 2 - 6 + 4 = 0$$
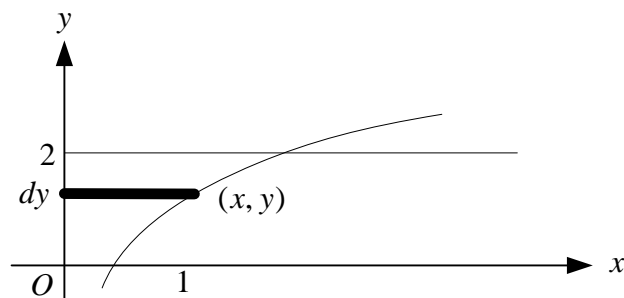
Ex-40-3: Fig-40-3 shows the curve $y = \ln x$ and part of the line $y = 2$.

Fig-40-3

The shaded region is rotated through 360^0 about the y-axis.

(i) Show that the volume of the solid of revolution formed is given by

$$\int_0^2 \pi e^{2y} dy.$$

$y = \ln x \implies x = e^y$

$$V = \int_0^2 \pi r^2 dy = \int_0^2 \pi x^2 dy = \int_0^2 \pi e^{2y} dy = \pi \int_0^2 e^{2y} dy$$

(ii) Evaluate this, leaving your answer in an exact form.

$$V = \int_0^2 \pi\, r^2 dy = \pi \int_0^2 e^{2y} dy = \frac{\pi}{2} \left(e^{2y}\right)\Big|_0^2 = \frac{\pi}{2}\left(e^4 - 1\right)$$

Ex-40-4:   A curve is defined by parametric equations

$$x = \frac{1}{t} - 1, \qquad y = \frac{2+t}{1+t}$$

Show that the Cartesian equation of the curve is $y = \dfrac{3+2x}{2+x}$

Ans:

$$x = \frac{1}{t} - 1 = \frac{1-t}{t} \quad \Rightarrow xt = 1-t \quad \Rightarrow xt + t = 1 \quad \Rightarrow t(x+1) = 1 \quad \Rightarrow t = \frac{1}{x+1}$$

$$y = \frac{2+t}{1+t} = \frac{2 + \dfrac{1}{x+1}}{1 + \dfrac{1}{x+1}} = \frac{\dfrac{2(x+1)+1}{x+1}}{\dfrac{x+1+1}{x+1}} = \frac{2x+2+1}{x+1+1} = \frac{2x+3}{x+2}$$

Ex-40-5:   Verify that the point (-1, 6, 5) lies on both the lines

$$r = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \qquad \text{and} \qquad r = \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

Ans:

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 - \lambda \\ 2 + 2\lambda \\ -1 + 3\lambda \end{pmatrix}$$

When $x = -1 = 1 - \lambda \Rightarrow \lambda = 2$

$\qquad\qquad y = 2 + 2\lambda = 2 + 2(2) = 2 + 4 = 6 \quad OK$

$\qquad\qquad z = -1 + 3\lambda = -1 + 3(2) = -1 + 6 = 5 \quad OK$

$$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} \mu \\ 6 \\ 3 - 2\mu \end{pmatrix}$$

When $x = -1 \Rightarrow \mu = -1$

$\qquad\qquad y = 6 \quad OK$

$\qquad\qquad z = 3 - 2\mu = 3 - 2(-1) = 3 + 2 = 5 \quad OK$

Find the acute angle between the lines.

Ans:

Angle between $\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ is θ, where

$\cos\theta = \dfrac{-1\times 1 + 2\times 0 + 3\times(-2)}{\sqrt{1+4+9}\times\sqrt{1+4}} = -\dfrac{7}{\sqrt{70}} \Rightarrow \theta = 146.8^0 \Rightarrow acute\ angle\ is\ 180 - 146.8 = 33.2^0$

Ex-40-6: Two students are trying to evaluate the integral $\displaystyle\int_1^2 \sqrt{1+e^{-x}}\,dx$.

Aisha uses the trapezium rule with 2 strips, and starts by constructing the following table.

x	1	1.5	2
$\sqrt{1+e^{-x}}$	1.1696	1.1060	1.0655

(i) Complete the calculation, giving your answer to 3 significant figures.

$h = \dfrac{b-a}{n} = \dfrac{2-1}{2} = 0.5$

$\Rightarrow \displaystyle\int_1^2 \sqrt{1+e^{-x}}\,dx \approx \dfrac{h}{2}(y_0 + 2y_1 + y_2) \approx \dfrac{0.5}{2}(1.1696 + 2\times(1.106) + 1.0655) \approx 1.11$

Ahmad uses a binomial approximation for and then integrates this.

(ii) Show that, provided e^{-x} is suitably small, $(1+e^{-x})^{\frac{1}{2}} \approx 1 + \dfrac{1}{2}e^{-x} - \dfrac{1}{8}e^{-2x}$.

Ans:

$(1+e^{-x})^{\frac{1}{2}} = 1 + \dfrac{1}{2}e^{-x} + \dfrac{\dfrac{1}{2}\left(-\dfrac{1}{2}\right)}{2!}(e^{-x})^2 + ...$

$(1+e^{-x})^{\frac{1}{2}} \approx 1 + \dfrac{1}{2}e^{-x} - \dfrac{1}{8}e^{-2x} + ...$

(iii) Use this result to evaluate $\displaystyle\int_1^2 \sqrt{1+e^{-x}}\,dx$ approximately, giving your answer to 3 significant figures.

Ans:

$$I = \int_1^2 \left(1 + \frac{1}{2}e^{-x} - \frac{1}{8}e^{-2x}\right) dx$$

$$I = \left[x - \frac{1}{2}e^{-x} + \frac{1}{16}e^{-2x} \right]_1^2 = \left[2 - \frac{1}{2}e^{-2} + \frac{1}{16}e^{-4} \right] - \left[1 - \frac{1}{2}e^{-1} + \frac{1}{16}e^{-2} \right]$$

$$I = 1.9335 - 0.8245 = 1.11 \ (3 \ S.F.)$$

SECTION B (P-40)

Ex-40-7: Data suggest that the number of cases of infection from a particular disease tends to oscillate between two values over a period of approximately 6 months.

(a) Suppose that the number of cases, P thousand, after time t months is modelled by the equation $P = \dfrac{2}{2 - \sin t}$. Thus, when $t = 0$, $P = 1$.

(i) By considering the greatest and least values of P, write down the greatest and least values of P predicted by this model.

Ans:

$$P_{max} = \frac{2}{2-1} = 2$$

$$P_{min} = \frac{2}{2+1} = \frac{2}{3}$$

(ii) Verify that P satisfies the differential equation $\dfrac{dP}{dt} = \dfrac{1}{2}P^2 \cos t$

Ans:

$$P = \frac{2}{2-\sin t} = 2(2-\sin t)^{-1}$$

$$\frac{dp}{dt} = -2(2-\sin t)^{-2}(-\cos t) = \frac{2\cos t}{(2-\sin t)^2}$$

$$\frac{1}{2}P^2 \cos t = \frac{1}{2}\frac{4}{(2-\sin t)^2}\cos t = \frac{2\cos t}{(2-\sin t)^2} = \frac{dp}{dt}$$

(b) An alternative model is proposed, with differential equation

$$\frac{dP}{dt} = \frac{1}{2}\left(2P^2 - P\right)\cos t$$

As before, $P = 1$ when $t = 0$.

(i) Express $\dfrac{1}{P(2P-1)}$ in partial fractions.

Ans:

$$\frac{1}{P(2P-1)} = \frac{1}{2P\left(P-\dfrac{1}{2}\right)} = \frac{A}{P} + \frac{B}{P-\dfrac{1}{2}}$$

$$A = \left.\frac{1}{2\left(P-\dfrac{1}{2}\right)}\right|_{P=0} = -1$$

$$B = \left.\frac{1}{2P}\right|_{P=\dfrac{1}{2}} = 1$$

$$\frac{1}{P(2P-1)} = \frac{1}{2P\left(P-\dfrac{1}{2}\right)} = -\frac{1}{P} + \frac{1}{P-\dfrac{1}{2}} = -\frac{1}{P} + \frac{2}{2P-1}$$

(ii) Solve the differential equation (*) to show that
$$\ln\left(\frac{2P-1}{P}\right) = \frac{1}{2}\sin t$$

Ans:

$$\frac{dP}{dt} = \frac{1}{2}\left(2P^2 - P\right)\cos t$$

$$\int \frac{dP}{2P^2 - P} = \frac{1}{2}\int \cos t\, dt$$

$$\int \left(\frac{2}{2P-1} - \frac{1}{P}\right) dp = \frac{1}{2}\int \cos t\, dt$$

$$\int \frac{2}{2P-1} dp - \int \frac{dp}{P} = \frac{1}{2}\int \cos t\, dt$$

$$\ln(2P-1) - \ln P = \frac{1}{2}\sin t + C$$

When t = 0, P=1

$$\ln(2-1) - \ln 1 = 0 + C \Rightarrow C = \ln 1 - \ln 1 = 0 - 0 = 0$$

$$\ln(2P-1) - \ln P = \frac{1}{2}\sin t + 0$$

$$\ln\left(\frac{2P-1}{P}\right) = \frac{1}{2}\sin t$$

This equation can be rearranged to give $P = \dfrac{1}{2 - e^{\frac{1}{2}\sin t}}$

(iii) Find the greatest and least values of P predicted by this model.

Ans:

$$P_{max} = \frac{1}{2 - e^{\frac{1}{2}}} = 2.847$$

$$P_{min} = \frac{1}{2 - e^{-\frac{1}{2}}} = 0.718$$

Ex-40-8: In a theme park ride, a capsule C moves in a vertical plane (see Fig40-8). With respect to the axes shown, the path of C is modelled by the parametric equations

$$x = 10\cos\theta + 5\cos 2\theta, \qquad y = 10\sin\theta + 5\sin 2\theta, \quad (0 \le \theta \le 2\pi)$$

where x and y are in metres.

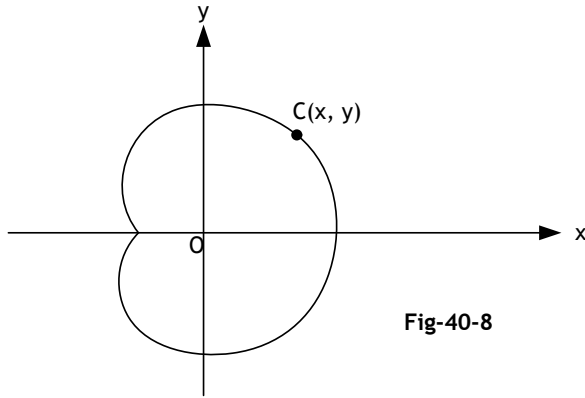Fig-40-8

(i) Show that $\dfrac{dy}{dx} = -\dfrac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$

$x = 10\cos\theta + 5\cos 2\theta$

$\dfrac{dx}{dt} = -10\sin\theta - 10\sin 2\theta$

$y = 10\sin\theta + 5\sin 2\theta$

$\dfrac{dy}{dt} = 10\cos\theta + 10\cos 2\theta$

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = \dfrac{10\cos\theta + 10\cos 2\theta}{-10\sin\theta - 10\sin 2\theta} = -\dfrac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$

Verify that $\dfrac{dy}{dx} = 0$ when $\theta = \dfrac{1}{3}\pi$. Hence find the exact coordinates of the highest point A on the path of C.

$\dfrac{dy}{dx} = -\dfrac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}\bigg|\theta = \dfrac{1}{3}\pi = -\dfrac{\cos\left(\dfrac{\pi}{3}\right) + \cos\left(\dfrac{2\pi}{3}\right)}{\sin\left(\dfrac{\pi}{3}\right) + \sin\left(\dfrac{2\pi}{3}\right)} = -\dfrac{\dfrac{1}{2} + \left(-\dfrac{1}{2}\right)}{\dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{3}}{2}} = -\dfrac{0}{\sqrt{3}} = 0$

$x = 10\cos\theta + 5\cos 2\theta\bigg|\theta = \dfrac{1}{3}\pi = 10\cos\left(\dfrac{\pi}{3}\right) + 5\cos\left(\dfrac{2\pi}{3}\right) = \dfrac{10}{2} - \dfrac{5}{2} = \dfrac{5}{2}$

$y = 10\sin\theta + 5\sin 2\theta\bigg|\theta = \dfrac{1}{3}\pi = 10\sin\left(\dfrac{\pi}{3}\right) + 5\sin\left(\dfrac{2\pi}{3}\right) = \dfrac{10\sqrt{3}}{2} + \dfrac{5\sqrt{3}}{2} = \dfrac{15\sqrt{3}}{2}$

Point $\left(\dfrac{5}{2}, \dfrac{15\sqrt{3}}{2}\right)$

(ii) Express $x^2 + y^2$ in terms of θ. Hence show that

$x^2 + y^2 = 125 + 100\cos\theta$

$x = 10\cos\theta + 5\cos 2\theta$

$$\Rightarrow x^2 = (10\cos\theta + 5\cos 2\theta)^2 = 100\cos^2\theta + 100\cos\theta\cos 2\theta + 25\cos^2 2\theta$$
$$y = 10\sin\theta + 5\sin 2\theta$$
$$\Rightarrow y^2 = (10\sin\theta + 5\sin 2\theta)^2 = 100\sin^2\theta + 100\sin\theta\sin 2\theta + 25\sin^2 2\theta$$
$$\Rightarrow x^2 + y^2 = 100\cos^2\theta + 100\cos\theta\cos 2\theta + 25\cos^2 2\theta + 100\sin^2\theta + 100\sin\theta\sin 2\theta + 25\sin^2 2\theta$$

$$\Rightarrow x^2 + y^2 = 100(\sin^2\theta + \cos^2\theta) + 25(\sin^2 2\theta + \cos^2 2\theta) + 100\cos\theta\cos 2\theta + 100\sin\theta\sin 2\theta$$
$$\Rightarrow x^2 + y^2 = 125 + 100\cos\theta\cos 2\theta + 100\sin\theta\sin 2\theta$$
$$\Rightarrow \cos 2\theta = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta = \cos^2\theta - \sin^2\theta$$
$$\Rightarrow \sin 2\theta = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta$$
$$\Rightarrow x^2 + y^2 = 125 + 100\cos\theta(\cos^2\theta - \sin^2\theta) + 200\sin\theta\sin\theta\cos\theta$$
$$\Rightarrow x^2 + y^2 = 125 + 100\cos\theta[\cos^2\theta - \sin^2\theta + 2\sin^2\theta]$$
$$\Rightarrow x^2 + y^2 = 125 + 100\cos\theta[\cos^2\theta + \sin^2\theta]$$
$$\Rightarrow x^2 + y^2 = 125 + 100\cos\theta$$

(iii) Using this result, or otherwise, find the greatest and least distances of C from O.

Ans:

$$Max\sqrt{125 + 100} = 15$$

$$Min\sqrt{125 - 100} = 5$$

You are given that, at the point B on the path vertically above O,
$$2\cos^2\theta + 2\cos\theta - 1 = 0$$

(iv) Using this result, and the result in part **(ii)**, find the distance OB. Give your answer to 3 significant figures.

Ans:

$$2\cos^2\theta + 2\cos\theta - 1 = 0$$

$$\cos\theta = \frac{-2 \pm \sqrt{12}}{4} = \frac{-2 \pm 2\sqrt{3}}{4} = \frac{-1 \pm \sqrt{3}}{2}$$

at B, $\cos\theta = \dfrac{-1 + \sqrt{3}}{2}$

$$OB^2 = 125 + 50(-1 + \sqrt{3}) = 75 + 50\sqrt{3} = 161.6$$

$$\Rightarrow OB = \sqrt{161.6} = 12.7\ (m)$$
$$2\cos^2\theta + 2\cos\theta - 1 = 0$$
$$\cos\theta = \frac{-2 \pm \sqrt{12}}{4} = \frac{-2 \pm 2\sqrt{3}}{4}$$

PAPER-41

Additional Information

$$ax^2 + bx + c = 0$$

Applications of Discriminate $\Delta = b^2 - 4ac$

(a) when $\Delta = b^2 - 4ac > 0$, there are TWO real distinct solutions

Ex-41-1: Solve this: $x^2 + 3x + 2 = 0$

$$\Delta = b^2 - 4ac = 3^2 - 4 \times 1 \times 2 = 9 - 8 = 1 > 0$$

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-3 + \sqrt{3^2 - 4(1)(2)}}{2(1)} = \frac{-3 + \sqrt{9 - 8}}{2} = \frac{-3 + 1}{2} = -1$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-3 - \sqrt{3^2 - 4(1)(2)}}{2(1)} = \frac{-3 - \sqrt{9 - 8}}{2} = \frac{-3 - 1}{2} = -2$$

(b) when $\Delta = b^2 - 4ac = 0$, there are TWO real equal solutions

Ex-41-2: Solve this: $x^2 + 6x + 9 = 0$

$$\Delta = \sqrt{b^2 - 4ac} = \sqrt{6^2 - 4 \times 1 \times 9} = \sqrt{36 - 36} = \sqrt{0} = 0$$

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-6 + \sqrt{6^2 - 4(1)(9)}}{2(1)} = \frac{-6 + \sqrt{36 - 36}}{2} = \frac{-6 + \sqrt{0}}{2} = \frac{-6}{2} = -3$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-6 - \sqrt{6^2 - 4(1)(9)}}{2(1)} = \frac{-6 - \sqrt{36 - 36}}{2} = \frac{-6 - \sqrt{0}}{2} = \frac{-6}{2} = -3$$

(c) when $\Delta = b^2 - 4ac < 0$, there are **NO** real solutions. The roots are complex numbers.

Ex-41-3: Solve this: $x^2 + 2x + 10 = 0$

$$\Delta = \sqrt{b^2 - 4ac} = \sqrt{2^2 - 4 \times 1 \times 10} = \sqrt{4 - 40} = \sqrt{-36} = ?$$

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-2+\sqrt{2^2-4(1)(10)}}{2(1)} = \frac{-2+\sqrt{4-40}}{2} = \frac{-2+\sqrt{-36}}{2} = \frac{-2+j6}{2} = -1+j3$$

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-2-\sqrt{2^2-4(1)(10)}}{2(1)} = \frac{-2-\sqrt{4-40}}{2} = \frac{-2-\sqrt{-36}}{2} = \frac{-2-j6}{2} = -1-j3$$

Application Examples:

Ex-41-4: (i) Find the range of values of k for which the equation
$x^2 + 5x + k = 0$ has one or more real roots.

Ans: For real roots: $b^2 - 4ac \geq 0$

$$5^2 - 4k \geq 0 \qquad -4k \geq -25 \qquad 4k \leq 25 \quad \Rightarrow k \leq \frac{25}{4}$$
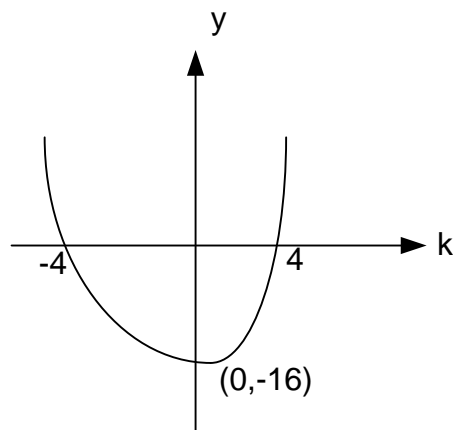
(ii) Solve the equation $4x^2 + 20x + 25 = 0$

$$\Rightarrow x_{1,2} = \frac{-b \pm \sqrt{b^2}}{2a} = \frac{-20 \pm \sqrt{(20)^2 - 4(4)(25)}}{2(4)} = \frac{-20 \pm \sqrt{400-400}}{8}$$

$$= \frac{-20 \pm \sqrt{0}}{8} = -\frac{20}{8} = -\frac{5}{2} \qquad \Rightarrow x_1 = x_2 = -\frac{5}{2}$$

Ex-41-5: Find the set of values of k for which the equation $2x^2 + kx + 2 = 0$ has no real roots.

Ans: The equation $2x^2 + kx + 2 = 0$ has no real roots when
$$b^2 - 4ac < 0 \quad \Rightarrow k^2 - 4(2)(2) < 0 \qquad \Rightarrow k^2 - 16 < 0 \quad \Rightarrow (k-4)(k+4) < 0$$

$$-4 < k < 4$$

(x − 4) (x + 4) < 0		(x + 4)		(x - 4)	
		x = - 4		x = 4	
(x - 4)	-		-	0	+
(x + 4)	-	0	+		+
(x − 4)(x + 4)	+		-		+

Ans: - 4 < x < 4

(ii)   Find the set of values of k for which the equation $2x^2 + 3x - k = 0$ has no real roots.

The equation $2x^2 + 3x - k = 0$ has no real roots when $b^2 - 4ac < 0$, hence

$$3^2 - 4(2)(-k) < 0 \Rightarrow 8k < -9 \Rightarrow k < -\frac{9}{8}$$

Ex-41-6:   Find the range of values of k for which the equation $2x^2 + kx + 18 = 0$ does not have real roots.

Ans: The equation $2x^2 + kx + 18 = 0$ has no real roots when

$$b^2 - 4ac < 0 \Rightarrow k^2 - 4 \times 2 \times 18 < 0 \Rightarrow k^2 - 144 < 0 \Rightarrow k^2 < 144$$

$$-12 < k < 12$$

Ex-41-7:   Find the set of values of k for which the graph of $y = x^2 + 2kx + 5$ does not intersect the x-axis.

Ans: The equation $y = x^2 + 2kx + 5$ has no real roots when

$$b^2 - 4ac < 0 \Rightarrow (2k)^2 - 4 \times 5 < 0 \Rightarrow 4k^2 - 20 < 0$$

$$\Rightarrow k^2 - 5 < 0 \Rightarrow k^2 < 5 \Rightarrow -\sqrt{5} < k < \sqrt{5}$$

Ex-41-8:   Find the range of values of k for which the equation $x^2 + 5x + k = 0$ has one or more real roots.

Ans:

$$b^2 - 4ac \geq 0 \Rightarrow (5)^2 - 4 \times k \geq 0 \Rightarrow -4k \geq -25 \Rightarrow k \leq \frac{25}{4}$$

Ex-41-9: Show that the graph of $y = x^2 - 3x + 11$ is above the x-axis for all values of x.
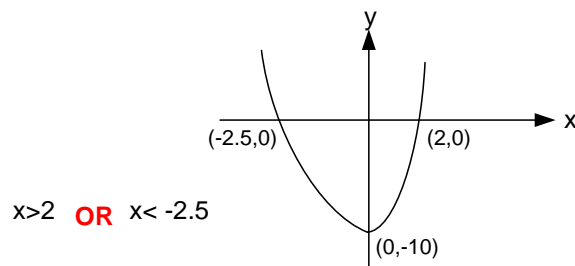
Ans:

$$y = x^2 - 3x + 11 = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 11 = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{44}{4} = \left(x - \frac{3}{2}\right)^2 + \frac{35}{4}$$

Ex-41-10: Find the set of values of x for which the graph of $y = 2x^2 + x - 10$ is above the x-axis.
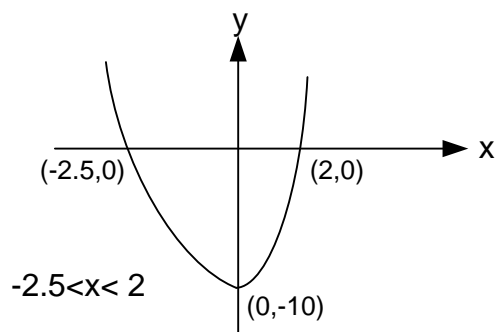
Ans:

The graph will be above x-axis when $y > 0$, hence $y = 2x^2 + x - 10 > 0$

$$(2x + 5)(x - 2) > 0$$

x>2 **OR** x< -2.5

Ex-41-11: Find the set of values of x for which the graph of $y = 2x^2 + x - 10$ is below the x-axis.

Ans:

The graph will be below x-axis when $y < 0$, hence $y = 2x^2 + x - 10 < 0$

$$(2x + 5)(x - 2) < 0$$

-2.5<x< 2

PAPER-42

Amplitude

Ex-42-1: $\qquad f(t) = \sin 5t + \cos 5t$ $\qquad\qquad\qquad$ ...(1)

$$f(t) = \sin 5t + \cos 5t = A\cos(5t + \theta) = A\cos 5t \cos\theta - A\sin 5t \sin\theta$$

$$\Rightarrow f(t) = -(A\sin\theta)\sin 5t + (A\cos\theta)\cos 5t \qquad\qquad ...(2)$$

Comparing Eqs 1 *and* 2:

$$\Rightarrow -A\sin\theta = 1 \qquad and$$
$$A\cos\theta = 1$$

$$\Rightarrow \frac{-A\sin\theta}{A\cos\theta} = -\tan\theta = \frac{1}{1} = 1$$

$$\Rightarrow \tan\theta = -1 \quad \Rightarrow \theta = -45^0$$

$$\Rightarrow A = \frac{1}{\cos(-45^0)} = \sqrt{2}$$

$$\Rightarrow f(t) = \sin 5t + \cos 5t = \sqrt{2}\cos(5t - 45^0)$$

Ex-42-2: $\qquad f(t) = \sin 5t + \cos(5t + 60^0)$

$$f(t) = \sin 5t + \cos(5t + 60^0) = \sin 5t + \cos 5t \cos 60^0 - \sin 5t \sin 60^0$$
$$\Rightarrow f(t) = \sin 5t + 0.5\cos 5t - 0.866\sin 5t = 0.1339\sin 5t + 0.5\cos 5t$$
$$\Rightarrow f(t) = 0.1339\sin 5t + 0.5\cos 5t = A\cos(5t + \theta) = A\cos\theta\cos 5t - A\sin\theta\sin 5t$$

$$\Rightarrow A\cos\theta = 0.5 \qquad and \quad -A\sin\theta = 0.1339 \qquad \tan\theta = -\frac{0.1339}{0.5}$$

$$\Rightarrow \theta = -15^0 \quad and \quad A = \frac{0.5}{\cos\theta} = \frac{0.5}{\cos(-15^0)} = 0.518 \quad \Rightarrow f(t) = 0.518\cos(5t - 15^0)$$

Ex-42-3: $\qquad f(t) = 2\sin 10t - 5\cos 10t$ $\qquad\qquad\qquad$ ...(1)

$$f(t) = 2\sin 10t - 5\cos 10t = A\sin(10t + \theta) = A\sin 10t \cos\theta + A\cos 10t \sin\theta$$

$$\Rightarrow f(t) = (A\cos\theta)\sin 10t + (A\sin\theta)\cos 10t \qquad\qquad ...(2)$$

Comparing Eqs 1 *and* 2:

$\Rightarrow A\cos\theta = 2 \quad$ *and*

$\quad A\sin\theta = -5$

$\Rightarrow \dfrac{A\sin\theta}{A\cos\theta} = \tan\theta = \dfrac{-5}{2} = -2.5$

$\Rightarrow \tan\theta = -2.5 \quad \Rightarrow \theta = -68.2^0$

$\Rightarrow A = \dfrac{2}{\cos(-68.2^0)} = 5.385 \ (4 \ S.F)$

$\Rightarrow f(t) = 2\sin 10t - 5\cos 10t = 5.385\sin(10t - 68.2^0)$

Ex-42-4: $\quad f(t) = 3\sin 8t + 4\cos(8t + 75^0)$

$f(t) = 3\sin 8t + 4\cos(8t + 75^0) = 3\sin 8t + 4\cos 8t\cos 75^0 - 4\sin 8t\sin 75^0$

$\Rightarrow f(t) = 3\sin 8t + 01.035\cos 8t - 3.864\sin 8t = -0.864\sin 8t + 1.035\cos 8t$

$\Rightarrow f(t) = -0.864\sin 8t + 1.035\cos 10t = A\cos(5t + \theta) = A\cos(8t + \theta)$

$= A\cos\theta\cos 8t - A\sin\theta\sin 8t$

$\Rightarrow A\cos\theta = 1.035 \quad$ *and* $\quad -A\sin\theta = -0.864 \quad \tan\theta = \dfrac{0.864}{1.035} = 0.835$

$\Rightarrow \theta = 39.85^0 \quad$ *and* $\quad A = \dfrac{1.035}{\cos\theta} = \dfrac{1.035}{\cos(39.85^0)} = 1.348 \quad \Rightarrow f(t) = 1.348\cos(8t + 39.85^0)$

PAPER-43

Completing The Square

Ex-43-1: Solve this: $ax^2 + bx + c = 0$ by completing the square.

$ax^2 + bx + c = 0$

$a\left(x^2 + \dfrac{b}{a}x + \dfrac{c}{a}\right) = 0$

$\left(x^2 + \dfrac{b}{a}x + \dfrac{c}{a}\right) = 0$

$\left(x^2 + \dfrac{b}{a}x + \left(\dfrac{b}{2a}\right)^2 - \left(\dfrac{b}{2a}\right)^2 + \dfrac{c}{a}\right) = 0$

$$\left(x+\frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

$$\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2} = 0$$

$$\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2} = 0$$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

$$\left(x+\frac{b}{2a}\right) = \pm\sqrt{\frac{b^2-4ac}{4a^2}} = \frac{\pm\sqrt{b^2-4ac}}{2a}$$

$$x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad\qquad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Ex-43-2:   Solve this: $x^2 + 3x + 2 = 0$

$$\Delta = \sqrt{b^2-4ac} = \sqrt{3^2-4\times1\times2} = \sqrt{9-8} = \sqrt{1} = 1 > 0$$

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} = \frac{-3+\sqrt{3^2-4(1)(2)}}{2(1)} = \frac{-3+\sqrt{9-8}}{2} = \frac{-3+1}{2} = -1$$

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-3-\sqrt{3^2-4(1)(2)}}{2(1)} = \frac{-3-\sqrt{9-8}}{2} = \frac{-3-1}{2} = -2$$

Ex-43-3:   Solve this: $x^2 + 3x + 2 = 0$ by completing the square method

$$x^2 + 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2 = 0$$

$$\left(x+\frac{3}{2}\right)^2 + 2 - \frac{9}{4} = 0$$

$$\left(x+\frac{3}{2}\right)^2 - \frac{1}{4} = 0$$

$$\left(x+\frac{3}{2}\right)^2 = \frac{1}{4}$$

$$\left(x+\frac{3}{2}\right)=\pm\sqrt{\frac{1}{4}}=\pm\frac{1}{2}$$

$$x_{1,2}=-\frac{3}{2}\pm\frac{1}{2}$$

$$x_1=-\frac{3}{2}+\frac{1}{2}=-\frac{2}{2}=-1$$

$$x_2=-\frac{3}{2}-\frac{1}{2}=-\frac{4}{2}=-2$$

Ex-43-4: Solve this: $x^2+6x+9=0$

$$\Delta=\sqrt{b^2-4ac}=\sqrt{6^2-4\times1\times9}=\sqrt{36-36}=\sqrt{0}=0$$

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}=\frac{-6+\sqrt{36-36}}{2}=\frac{-6+\sqrt{0}}{2}=\frac{-6+0}{2}=-3$$

$$x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{-6-\sqrt{36-36}}{2}=\frac{-6-\sqrt{0}}{2}=\frac{-6+0}{2}=-3$$

Ex-43-5: Solve this: $x^2+6x+9=0$ by completing the square method

$$x^2+6x+(3)^2-(3)^2+9=0$$
$$(x+3)^2-9+9=0$$
$$(x+3)^2=0$$
$$x_1=-3$$
$$x_2=-3$$

Ex-43-6: Solve this: $x^2+2x+10=0$

$$\Delta=\sqrt{b^2-4ac}=\sqrt{2^2-4\times1\times10}=\sqrt{4-40}=\sqrt{-36}=?$$

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}=\frac{-2+\sqrt{2^2-4(1)(10)}}{2(1)}=\frac{-2+\sqrt{4-40}}{2}=\frac{-2+\sqrt{-36}}{2}=\frac{-2+j6}{2}=-1+j3$$

$$x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{-2-\sqrt{2^2-4(1)(10)}}{2(1)}=\frac{-2-\sqrt{4-40}}{2}=\frac{-2-\sqrt{-36}}{2}=\frac{-2-j6}{2}=-1-j3$$

Ex-43-7: Solve this: $x^2 + 2x + 10 = 0$ by completing the square method

$$\left(x^2 + 2x + 1^2 - 1^2 + 10\right) = 0$$
$$\left(x+1\right)^2 + 9 = 0$$
$$\left(x+1\right)^2 = -9$$
$$\left(x+1\right)^2 = \sqrt{-9}$$
$$\left(x+1\right)^2 = \sqrt{-1}\sqrt{9} = \pm j3$$
$$x_1 = -1 + j3$$
$$x_2 = -1 - j3$$

Ex-43-8: Solve this: $3x^2 + 10x + 3 = 0$ by completing the square method

$$3\left(x^2 + \frac{10}{3}x + 1\right) = 0$$

$$x^2 + \frac{10}{3}x + 1 = 0$$

$$x^2 + \frac{10}{3}x + \left(\frac{10}{6}\right)^2 - \left(\frac{10}{6}\right)^2 + 1 = 0$$

$$\left(x + \frac{10}{6}\right)^2 + 1 - \frac{100}{36} = 0$$

$$\left(x + \frac{10}{6}\right)^2 - \frac{64}{36} = 0$$

$$\left(x + \frac{10}{6}\right)^2 = \frac{64}{36}$$

$$\left(x + \frac{10}{6}\right) = \pm\sqrt{\frac{64}{36}} = \pm\frac{8}{6}$$

$$x_{1,2} = -\frac{10}{6} \pm \frac{8}{6}$$

$$x_1 = -\frac{10}{6} + \frac{8}{6} = -\frac{2}{6} = -\frac{1}{3}$$

$$x_2 = -\frac{10}{6} - \frac{8}{6} = -\frac{18}{6} = -3$$

Ex-43-9: Solve this: $-3x^2 + 10x - 3 = 0$ by completing the square method

$$-3x^2 + 10x - 3 = 0$$
$$3x^2 - 10x + 3 = 0$$

$$3\left(x^2 - \frac{10}{3}x + 1\right) = 0$$

$$x^2 - \frac{10}{3}x + 1 = 0$$

$$x^2 - \frac{10}{3}x + \left(\frac{10}{6}\right)^2 - \left(\frac{10}{6}\right)^2 + 1 = 0$$

$$\left(x - \frac{10}{6}\right)^2 + 1 - \frac{100}{36} = 0$$

$$\left(x - \frac{10}{6}\right)^2 - \frac{64}{36} = 0$$

$$\left(x - \frac{10}{6}\right)^2 - \frac{64}{36} = 0$$

$$\left(x - \frac{10}{6}\right)^2 = \frac{64}{36}$$

$$\left(x - \frac{10}{6}\right) = \pm\sqrt{\frac{64}{36}} = \pm\frac{8}{6}$$

$$x_{1,2} = \frac{10}{6} \pm \frac{8}{6}$$

$$x_1 = \frac{10}{6} + \frac{8}{6} = \frac{18}{6} = 3$$

$$x_2 = \frac{10}{6} - \frac{8}{6} = -\frac{2}{6} = -\frac{1}{3}$$

Ex-43-10: Solve this: $10x^2 - 7x + 1 = 0$ by completing the square method

$$10\left(x^2 - \frac{7}{10}x + \frac{1}{10}\right) = 0$$

$$x^2 - \frac{7}{10}x + \frac{1}{10} = 0$$

$$x^2 - \frac{7}{10}x + \left(\frac{7}{20}\right)^2 - \left(\frac{7}{20}\right)^2 + \frac{1}{10} = 0$$

$$\left(x - \frac{7}{20}\right)^2 + \frac{1}{10} - \frac{49}{400} = 0$$

$$\left(x - \frac{7}{20}\right)^2 - \frac{9}{400} = 0$$

$$\left(x - \frac{7}{20}\right) = \pm\sqrt{\frac{9}{400}} = \pm\frac{3}{20}$$

$$x_{1,2} = \frac{7}{20} \pm \frac{3}{20}$$

$$x_1 = \frac{7}{20} + \frac{3}{20} = \frac{10}{20} = \frac{1}{2}$$

$$x_2 = \frac{7}{20} - \frac{3}{20} = \frac{4}{20} = \frac{1}{5}$$

Ex-43-11: Solve this: $6 - x - x^2 = 0$ by completing the square method

$$6 - x - x^2 = 0$$

$$-x^2 - x + 6 = 0$$

$$-\left(x^2 + x - 6\right) = 0$$

$$x^2 + x - 6 = 0$$

$$\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 6 = 0$$

$$\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{24}{4} = 0$$

$$\left(x + \frac{1}{2}\right)^2 = \frac{25}{4}$$

$$\left(x + \frac{1}{2}\right) = \pm\sqrt{\frac{25}{4}} = \pm\frac{5}{2}$$

$$x_{1,2} = \pm\frac{5}{2} - \frac{1}{2}$$

$$x_1 = \frac{5}{2} - \frac{1}{2} = \frac{4}{2} = 2$$

$$x_2 = -\frac{5}{2} - \frac{1}{2} = -\frac{6}{2} = -3$$

Ex-43-12: Solve this: $4x^2 - x - 3 = 0$ by completing the square method

$$4x^2 - x - 3 = 0$$

$$4\left(x^2 - \frac{1}{4}x - \frac{3}{4}\right) = 0$$

$$x^2 - \frac{1}{4}x - \frac{3}{4} = 0$$

$$\left(x - \frac{1}{8}\right)^2 - \left(\frac{1}{8}\right)^2 - \frac{3}{4} = 0$$

$$\left(x - \frac{1}{8}\right)^2 - \frac{1}{64} - \frac{48}{64} = 0$$
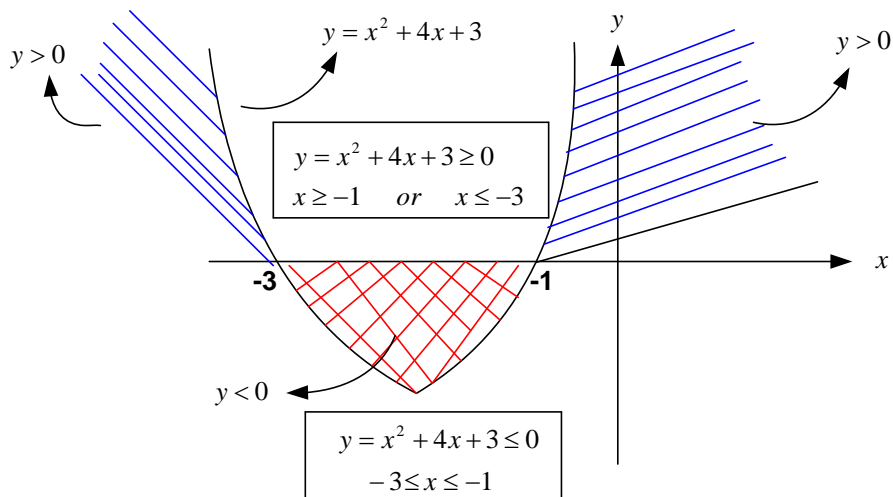
$$\left(x - \frac{1}{8}\right)^2 = \frac{49}{64}$$

$$\left(x - \frac{1}{8}\right) = \pm\sqrt{\frac{49}{64}} = \pm\frac{7}{8}$$

$$x_{1,2} = \pm\frac{7}{8} + \frac{1}{8}$$

$$x_1 = \frac{7}{8} + \frac{1}{8} = \frac{8}{8} = 1$$

$$x_2 = -\frac{7}{8} + \frac{1}{8} = -\frac{6}{8} = -\frac{3}{4}$$

$y > 0$

$y = x^2 + 4x + 3$

y

$y > 0$

$x^2 + 4x + 3 \geq 0$

$(x+3)(x+1) \geq 0$

$(x+3)(x+1) = 0$

$x = -3$

$x = -1$

$$\boxed{\begin{array}{l} y = x^2 + 4x + 3 \geq 0 \\ x \geq -1 \quad or \quad x \leq -3 \end{array}}$$

-3   **-1**

x

$y < 0$

$$\boxed{\begin{array}{l} y = x^2 + 4x + 3 \leq 0 \\ -3 \leq x \leq -1 \end{array}}$$

Ex-43-13:   Solve the inequality $x^2 + 4x + 3 > 0$

Ans:
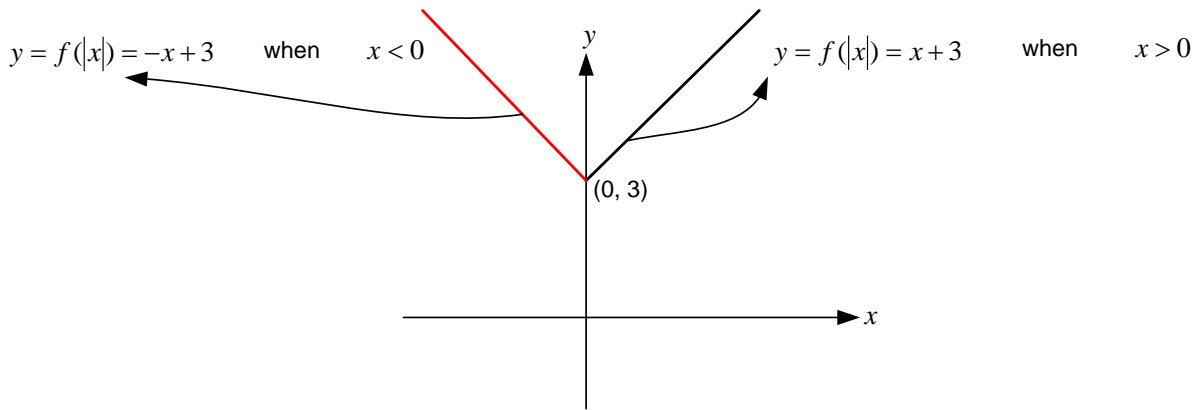
$$x^2 + 4x + 3 > 0$$
$$(x+3)(x+1) > 0$$

(x + 3) (x + 1) > 0   (x + 3)   (x + 1)

                          X = - 3   **X = - 1**

	-		**-**	**0**	**+**
	-	**0**	**+**		**+**
(x + 3) (x + 1)	**+**		**-**		**+**

                 Ans: **- 3 > x**   **or**   **x > - 1**

$$y = -x^2 - 6x - 5 \geq 0$$

(-3, 4)

$$\boxed{\begin{array}{l} y = -x^2 - 6x - 5 \geq 0 \\ -5 \leq x \leq -1 \end{array}}$$

$$y = -x^2 - 6x - 5 < 0$$

-5

-1

$$\boxed{\begin{array}{l} y = -x^2 - 6x - 5 < 0 \\ x < -5 \quad or \quad x > -1 \end{array}}$$

$$y = -x^2 - 6x - 5$$
$$y = -\left(x^2 + 6x + 5\right)$$

$$y = -\left[(x+3)^2 - 9 + 5\right]$$
$$y = -\left[(x+3)^2 - 4\right]$$
$$y = -(x+3)^2 + 4$$

$$y = -(x+3)^2 + 4 = 0$$
$$-(x+3)^2 = -4$$
$$(x+3)^2 = 4$$
$$(x+3) = \pm 2$$

$$x = -1$$
$$x = -5$$
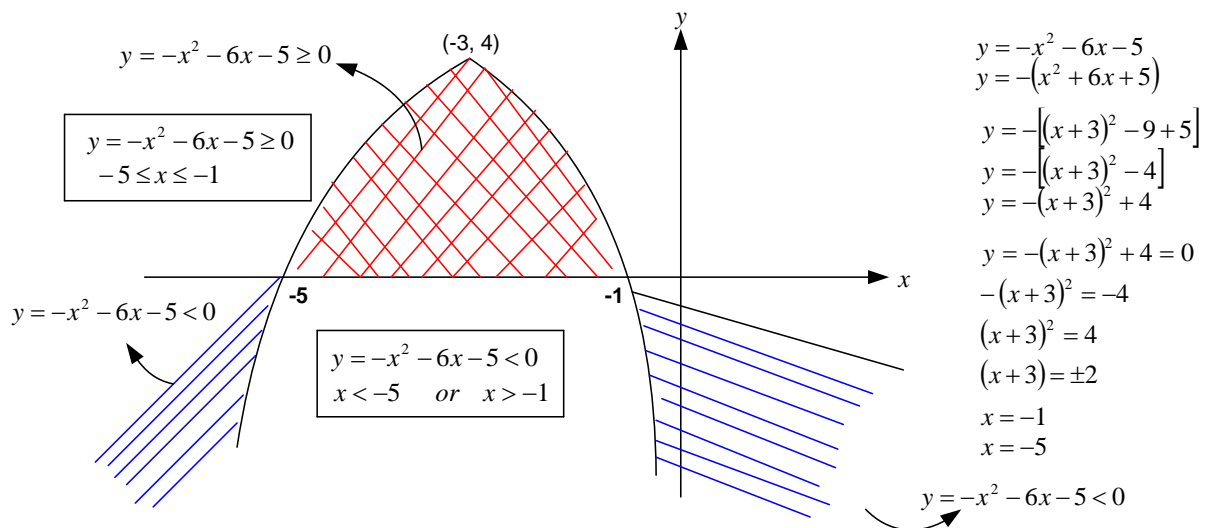
$$y = -x^2 - 6x - 5 < 0$$

Ex-43-14: Solve the inequality $y = -x^2 - 6x - 5 < 0$

Ans:

$$-x^2 - 6x - 5 < 0$$
$$x^2 + 6x + 5 > 0$$
$$(x+5)(x+1) > 0$$

(x + 5) (x + 1) > 0   (x + 5)   (x + 1)

x = - 5   x = - 1

-	-	0 +
-	0 +	+

(x + 5) (x + 1)   **+**   **-**   **+**

Ans: **- 5 > x   or   x > - 1**

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$y = f(2x) = 4x^2$
$y = 4x^2$

$y = f(x) = x^2$
$y = x^2$

$y = f\left(\dfrac{1}{2}x\right) = \dfrac{1}{4}x^2$
$y = \dfrac{1}{4}x^2$

(-1,4)　　(1,4)

(-1,1)　　(1,1)

(-1, 0.25)　　(1, 0.25)

(0,0)

-4　-3　-2　-1　1　2　3　4

x	y
0	3
3	0

x	y
0	3
-3	0

$y = f(x) = x + 3$

$y = f(-x) = -x + 3$

(0, 3)

(- 3, 0)　　(3, 0)

x	y
0	3
3	0

x	y
0	- 3
-3	0

$y = f(x) = x + 3$

(0, 3)

(- 3, 0)

$y = -f(x) = -x - 3$

(0, - 3)

$$y = f(|x|) = |x| + 3$$

$$y = f(|x|) = -x + 3 \quad \text{when} \quad x < 0$$
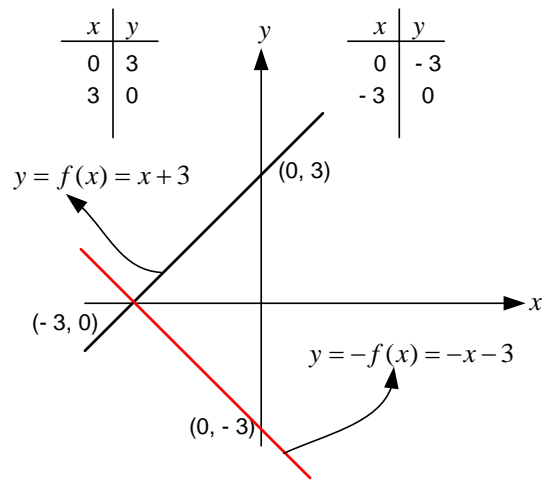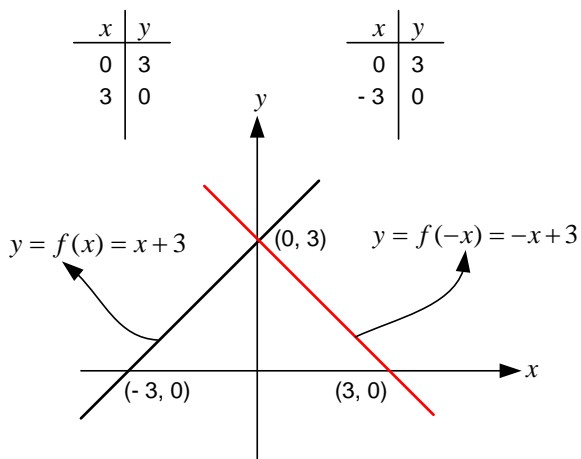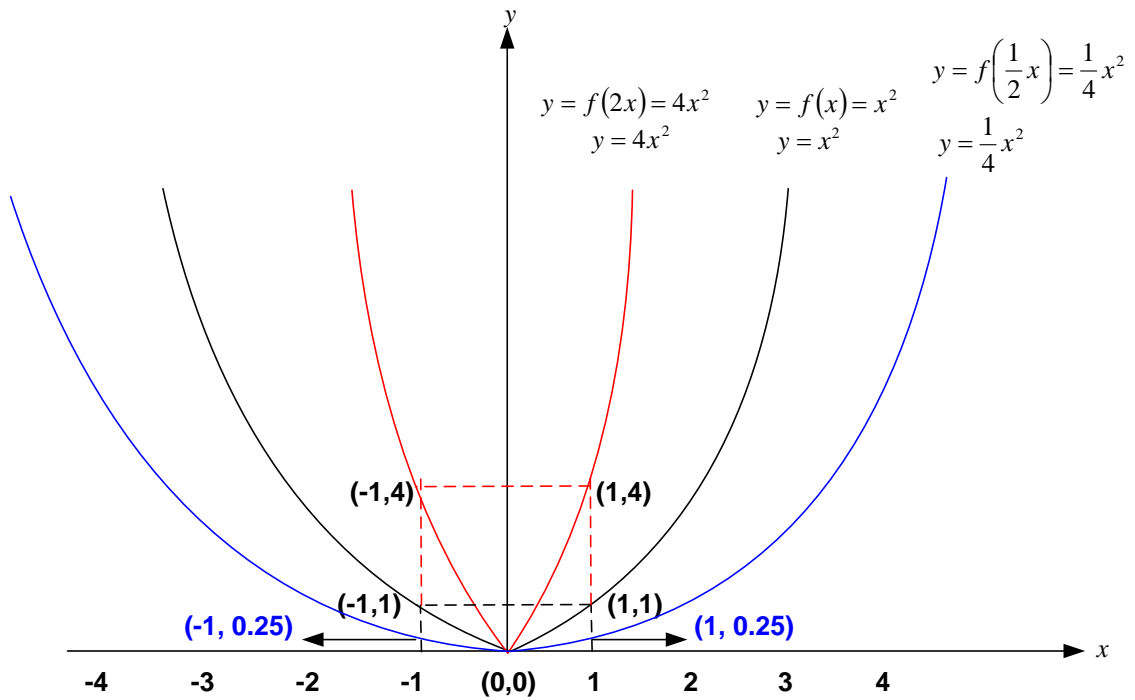
$$y = f(|x|) = x + 3 \quad \text{when} \quad x > 0$$

(0, 3)

$|x| = x$ when $x > 0$ $\Rightarrow y = f(|x|) = x + 3$ when $x > 0$

$|x| = -x$ when $x < 0$ $\Rightarrow y = f(|x|) = -x + 3$ when $x < 0$
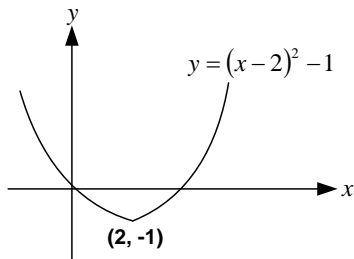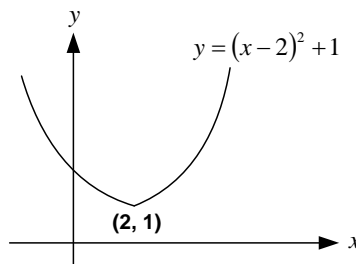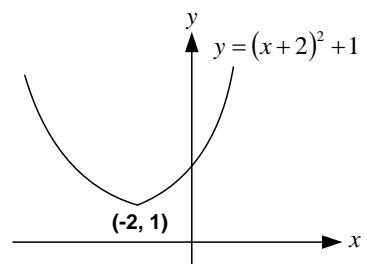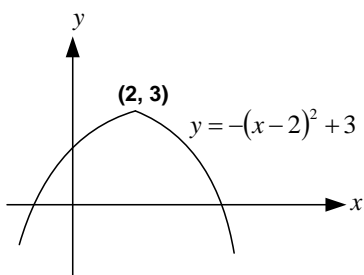
$y = (x-2)^2 - 1$

(2, -1)

Line of symmetry is x = 2

$y = (x-2)^2 + 1$

(2, 1)

Line of symmetry is x = 2

$y = (x+2)^2 + 1$

(-2, 1)

Line of symmetry is x = - 2

(2, 3)

$y = -(x-2)^2 + 3$

Line of symmetry is x = 2

(2, -2)

$y = -(x-2)^2 - 2$

Line of symmetry is x = 2

(-2, -1)

$y = -(x+2)^2 - 1$

Line of symmetry is x = - 2

$x = (y-1)^2 - 2$

(-2, 1)

Line of symmetry is y = 1

$x = (y+2)^2 + 2$

(2, - 2)

Line of symmetry is y = - 2

$x = (y-2)^2 + 2$

(2, 2)

Line of symmetry is y = 2

$x = -(y-2)^2 - 1$

(-1, 2)
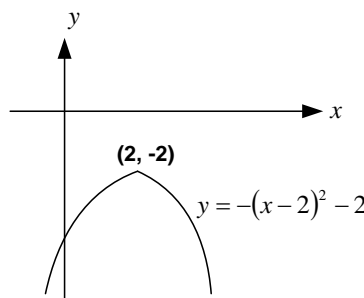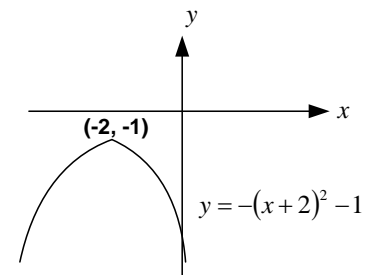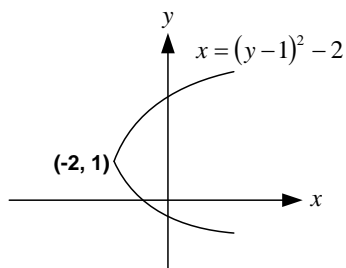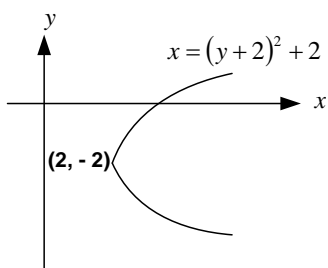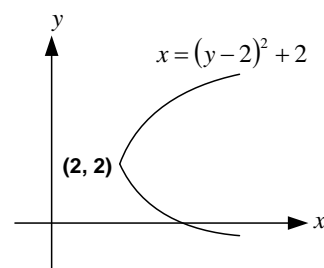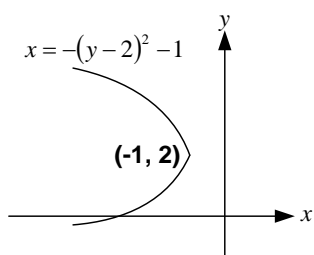
Line of symmetry is x-axis or y = 2

$x = -(y+1)^2 - 1$

(-1, -1)

Line of symmetry is x-axis or y = - 1

$x = -(y-1)^2 + 2$

(2, 1)
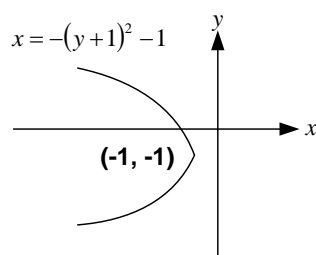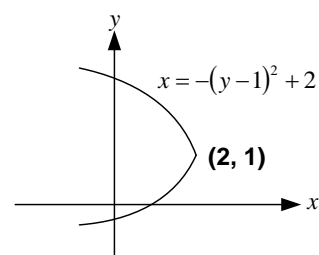
Line of symmetry is x-axis or y = 1

PAPER-44

Long Division and its Applications

Ex-44-1:   Divide $\dfrac{z}{z^2 - 3z + 2}$ using long division

$$
\begin{array}{r}
z^{-1} + 3z^{-2} + 7z^{-3} + 15z^{-4} \\
\hline
z^2 - 3z + 2 \,\big|\, z \\
\underline{-z + 3 - 2z^{-1}} \\
3 - 2z^{-1} \\
\underline{-3 + 9z^{-1} - 6z^{-2}} \\
7z^{-1} - 6z^{-2} \\
\underline{-7z^{-1} + 21z^{-2} - 14z^{-3}} \\
15z^{-2} - 14z^{-3} \\
\underline{-15z^{-2} + 45z^{-3} - 30z^{-4}} \\
31z^{-3} - 30z^{-4}
\end{array}
$$

Ex-44-2: Divide $\dfrac{x}{x^2 - 3x + 2}$ using long division.

$$
\begin{array}{r}
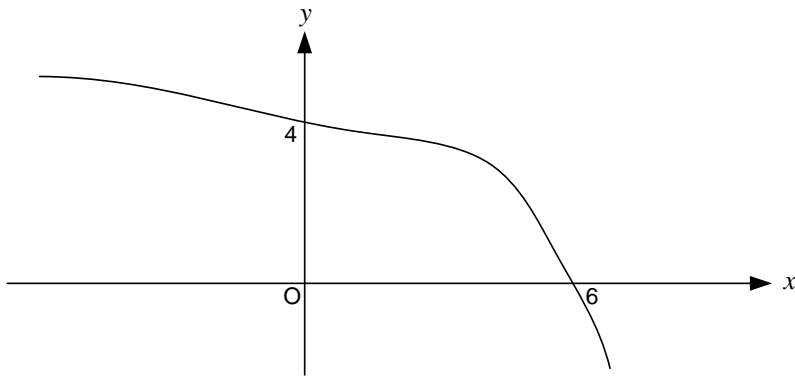x^{-1} + 3x^{-2} + 7x^{-3} + 15x^{-4} \\
\hline
\end{array}
$$

Long division working:

$x^2 - 3x + 2$)

x

$x - 3 + 2x^{-1}$
$\;-\;\;+\;\;\;-$
_____

$3 - 2x^{-1}$
$3 - 9x^{-1} + 6x^{-2}$
$\;-\;\;+\;\;\;\;-$
_____

$7x^{-1} - 6x^{-2}$
$7x^{-1} - 21x^{-2} + 14x^{-3}$
$\;-\;\;\;+\;\;\;\;\;-$
_____

$15x^{-2} - 14x^{-3}$
$15x^{-2} - 45x^{-3} + 30x^{-4}$
$\;-\;\;\;\;\;+\;\;\;\;\;-$
_____

$31x^{-3} - 30x^{-4}$

$$
\begin{array}{r}
x^3 + 2x^2 + 4x + 2 \\
\hline
\end{array}
$$

$x - 2$) $x^4 - 4x - 8$

$x^4 - 2x^3$
$-\quad+$
_____

$2x^3 - 4x$
$2x^3 - 4x^2$
$-\quad+$
_____

$4x^2 - 4x$
$4x^2 - 8x$
$-\quad+$
_____

$4x - 8$
$4x - 8$
$-\quad+$
_____

0

PAPER-45

Simpson's Rules + Others …

Ex-45-1: The diagram shows a sketch of the curve with equation $y = f(x)$.
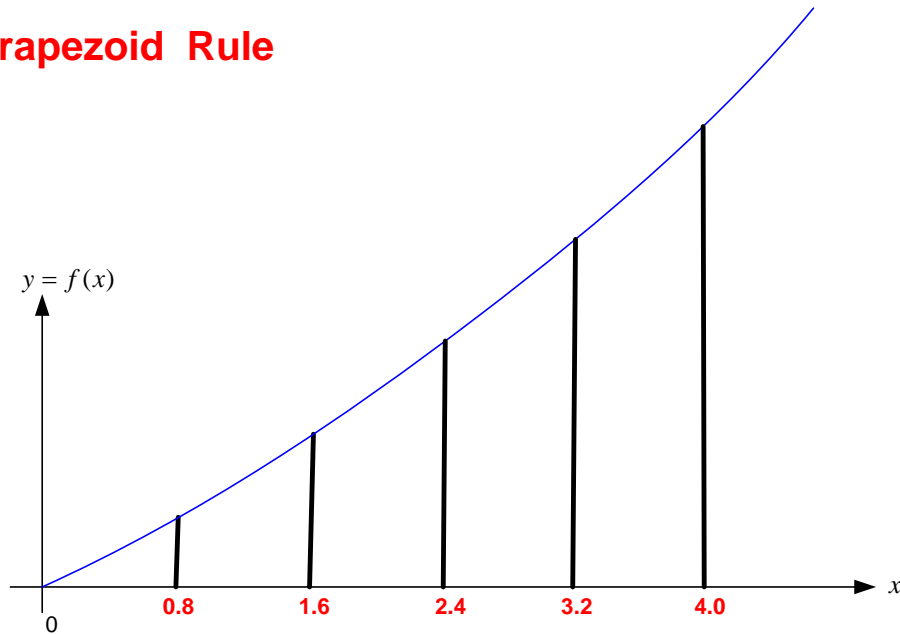
(a)   On Fig.1, below, sketch the curve with equation $y = -f(3x)$, indicating the values where the curve cuts the coordinate axes.

(b)   On Fig.2, sketch the curve with equation $y = f(|x|)$.

(c)   Describe the sequence of two geometrical transformations that maps the graph of $y = f(x)$ onto the graph of $y = f(-\dfrac{1}{2}x)$

Trapezoid Rule

$y = f(x)$
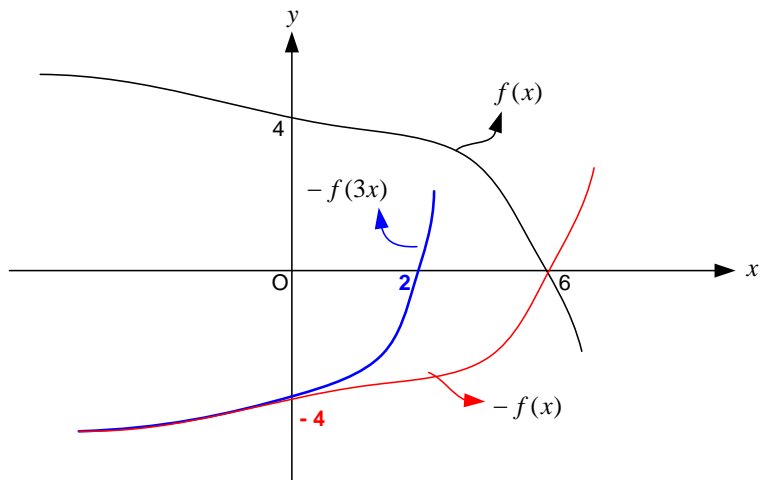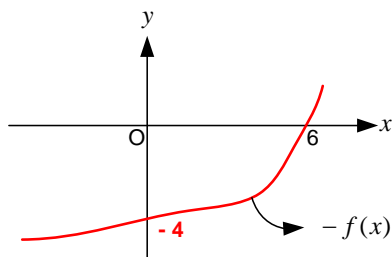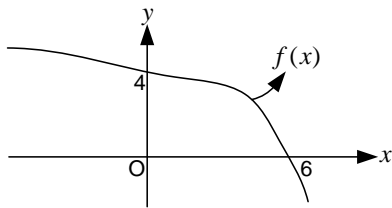
$a = 0, \quad b = 4$

$$h = \frac{b-a}{n} = \frac{4-0}{5} = 0.8$$

$$y = f(x) = \sqrt{27 + x^3}$$

$$\int_a^b f(x)dx = \int_0^4 \sqrt{27 + x^3} \cong \frac{h}{2}\left[f(a) + f(b) + 2\{f(a+h) + f(a+2h) + f(a+3h) + f(a+(n-1)h)\}\right]$$

$$\int_a^b f(x)dx = \int_0^4 \sqrt{27 + x^3} \cong \frac{h}{2}\left[f(a) + f(b) + 2\{f(a+h) + f(a+2h) + f(a+3h) + f(a+4h)\}\right]$$

$$f(x) = \sqrt{27 + x^3}$$

$$f(a) = f(0) = \sqrt{27 + (0)^3} = \sqrt{27} = 5.1961524$$

$$f(a+h) = f(0+0.8) = \sqrt{27 + (0.8)^3} = \sqrt{27 + 0.512} = 5.2451883$$

$$f(a+2h) = f(0+1.6) = \sqrt{27 + (1.6)^3} = \sqrt{27 + 4.096} = 5.5763788$$

$$f(a+3h) = f(0+2.4) = \sqrt{27 + (2.4)^3} = \sqrt{27 + 13.824} = 6.3893662$$

$$f(a+4h) = f(0+3.2) = \sqrt{27 + (3.2)^3} = \sqrt{27 + 32.768} = 7.7309767$$

$$f(a+5h) = f(0+4) = f(b) = \sqrt{27 + (4)^3} = \sqrt{27 + 64} = 9.539392$$

$$\int_a^b f(x)dx = \int_0^4 \sqrt{27 + x^3} \cong \frac{h}{2}\left[f(a) + f(b) + 2\{f(a+h) + f(a+2h) + f(a+3h) + f(a+4h)\}\right]$$

$$\int_a^b f(x)dx = \int_0^4 \sqrt{27 + x^3} \cong \frac{0.8}{2}\left[5.1961524 + 9.539392 + 2\{5.2451883 + 5.5763788 + 6.3893662 + 7.7309767\}\right]$$

$$\int_a^b f(x)dx = \int_0^4 \sqrt{27 + x^3} \cong 25.847746 \cong 25.8 \ (3 s.f)$$

Mid-Ordinate Rule
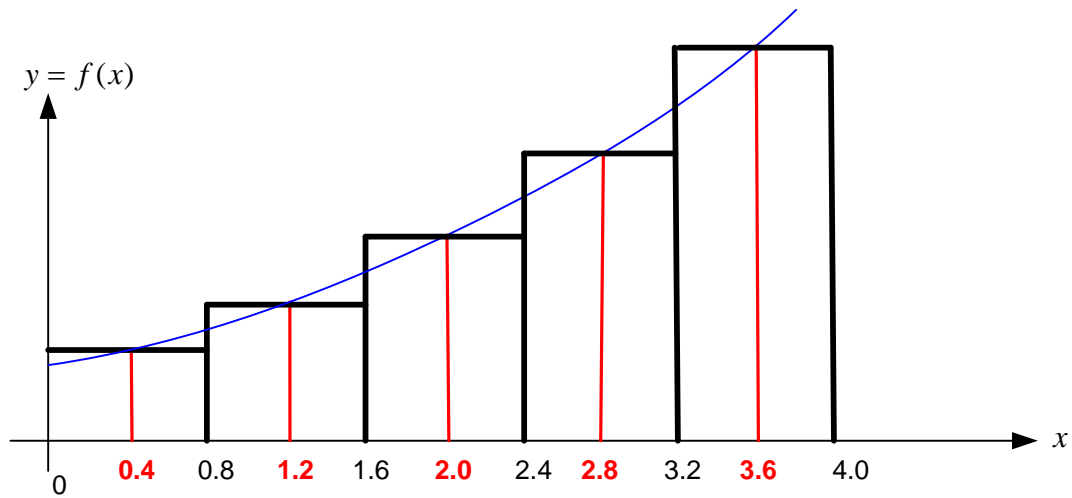
$$h = \frac{b-a}{n} = \frac{4-0}{5} = 0.8$$

$$y = f(x) = \sqrt{27 + x^3}$$

$$\int_a^b f(x)dx = \int_0^4 \sqrt{27 + x^3} \cong h[f(0.4) + f(1.2) + f(2) + f(2.8) + f(3.6)]$$

$$f(x) = \sqrt{27 + x^3}$$

$$f(0.4) = \sqrt{27 + (0.4)^3} = \sqrt{27 + 0.064} = 5.2023072$$

$$f(1.2) = \sqrt{27 + (1.2)^3} = \sqrt{27 + 1.728} = 5.3598507$$

$$f(2) = \sqrt{27 + (2)^3} = \sqrt{27 + 8} = 5.9160798$$

$$f(2.8) = \sqrt{27 + (2.8)^3} = \sqrt{27 + 21.952} = 6.9965706$$

$$f(3.6) = \sqrt{27 + (3.6)^3} = \sqrt{27 + 46.656} = 8.5823074$$

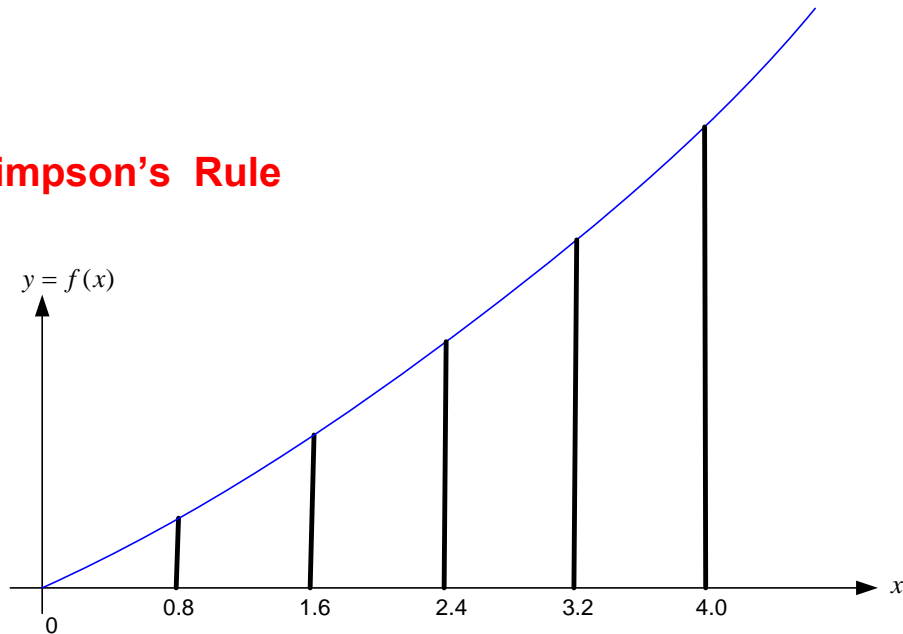$$\int_0^4 \sqrt{27 + x^3} \cong 0.8[5.2023072 + 5.3598507 + 5.9160798 + 6.9965706 + 8.5823074]$$

$$\int_0^4 \sqrt{27 + x^3} \cong 0.8(32.057116) = 25.645693 \cong 25.6 \ (3 \, s.f)$$

Simpson's Rule

$y = f(x)$

$a = 0, \quad b = 4$

$$h = \frac{b-a}{n} = \frac{4-0}{5} = 0.8$$

$$y = f(x) = \sqrt{27 + x^3}$$

$$\int_a^b f(x)dx = \int_0^4 \sqrt{27 + x^3} \cong \frac{h}{3}\left[f(a) + f(b) + 4\{f(a+h) + f(a+3h) + ...\} + 2\{f(a+2h) + f(a+4h) + ...\} \right]$$

$$f(x) = \sqrt{27 + x^3}$$

$$f(a) = f(0) = \sqrt{27 + (0)^3} = \sqrt{27} = 5.1961524$$

$$f(a+h) = f(0+0.8) = \sqrt{27 + (0.8)^3} = \sqrt{27 + 0.512} = 5.2451883$$

$$f(a+2h) = f(0+1.6) = \sqrt{27 + (1.6)^3} = \sqrt{27 + 4.096} = 5.5763788$$

$$f(a+3h) = f(0+2.4) = \sqrt{27 + (2.4)^3} = \sqrt{27 + 13.824} = 6.3893662$$

$$f(a+4h) = f(0+3.2) = \sqrt{27 + (3.2)^3} = \sqrt{27 + 32.768} = 7.7309767$$

$$f(a+5h) = f(0+4) = f(b) = \sqrt{27 + (4)^3} = \sqrt{27 + 64} = 9.539392$$

$$\int_a^b f(x)dx = \int_0^4 \sqrt{27 + x^3} \cong \frac{h}{3}\left[f(a) + f(b) + 4\{f(a+h) + f(a+3h) + ...\} + 2\{f(a+2h) + f(a+4h) + ...\} \right]$$

$$\int_a^b f(x)dx = \int_0^4 \sqrt{27 + x^3} \cong \frac{0.8}{3}\left[5.1961524 + 9.539392 + 4\{5.2451883 + 6.3893662\} + 2\{5.5763788 + 7.7309767\} \right]$$

$$\int_a^b f(x)dx = \int_0^4 \sqrt{27 + x^3} \cong \frac{0.8}{3}\left[14.735544 + 4\{11.634555\} + 2\{13.307356\} \right] \cong 23.436927 \cong 23.4 \ (3 \, s.f)$$

Trapezoid Rule

$a = 0, \quad b = \pi$
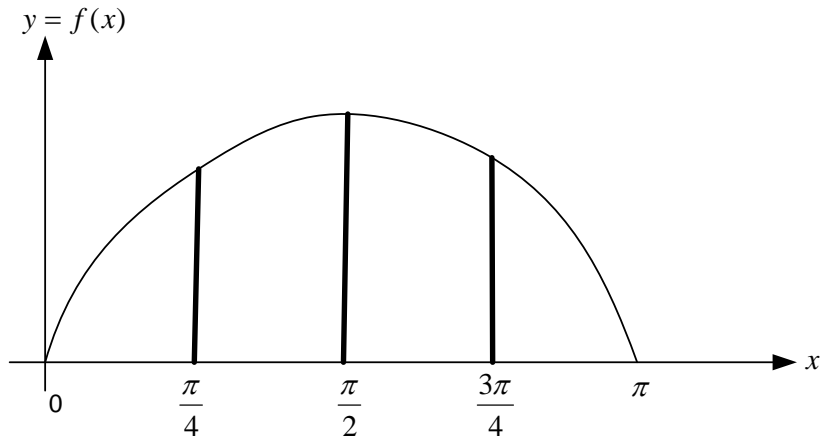
$$h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

$$y = f(x) = \sqrt{x} \, \sin x$$

$$\int_a^b f(x)dx = \int_0^\pi \sqrt{x} \sin x dx \cong \frac{h}{2}[f(a) + f(b) + 2\{f(a+h) + f(a+2h) + f(a+3h) + ...\}]$$

$$y = f(x) = \sqrt{x} \, \sin x$$

$$f(a) = f(0) = \sqrt{x} \, \sin x = \sqrt{0} \, \sin 0 = 0$$

$$f(a+h) = f\left(0 + \frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = \sqrt{\frac{\pi}{4}} \, \sin\left(\frac{\pi}{4}\right) = 0.8862269 \times 0.7071067 = 0.626657$$

$$f(a+2h) = f\left(0 + \frac{2\pi}{4}\right) = f\left(\frac{\pi}{2}\right) = \sqrt{\frac{\pi}{2}} \, \sin\left(\frac{\pi}{2}\right) = 1.2533141 \times 1 = 1.12533141$$

$$f(a+3h) = f\left(0 + \frac{3\pi}{4}\right) = f\left(\frac{3\pi}{4}\right) = \sqrt{\frac{3\pi}{4}} \, \sin\left(\frac{3\pi}{4}\right) = 1.5349901 \times 0.7071067 = 1.0854019$$

$$f(a+4h) = f\left(0 + \frac{4\pi}{4}\right) = f\left(\frac{4\pi}{4}\right) = f(\pi) = f(b) = \sqrt{\pi} \, \sin(\pi) = 1.7724539 \times 0 = 0$$

$$\int_a^b f(x)dx = \int_0^\pi \sqrt{x} \sin x dx \cong \frac{h}{2}[f(a) + f(b) + 2\{f(a+h) + f(a+2h) + f(a+3h) + ...\}]$$

$$\int_a^b f(x)dx = \int_0^\pi \sqrt{x} \sin x dx \cong \frac{\pi}{4 \times 2}[0 + 0 + 2\{0.626657 + 1.12533141 + 1.0854019\}]$$

$$\int_a^b f(x)dx = \int_0^\pi \sqrt{x} \sin x dx \cong \frac{\pi}{4 \times 2}[5.6747806] = 2.2284811 \cong 2.23 \; (3 \; s.f)$$

Mid-Ordinate Rule

$a = 0, \quad b = \pi$
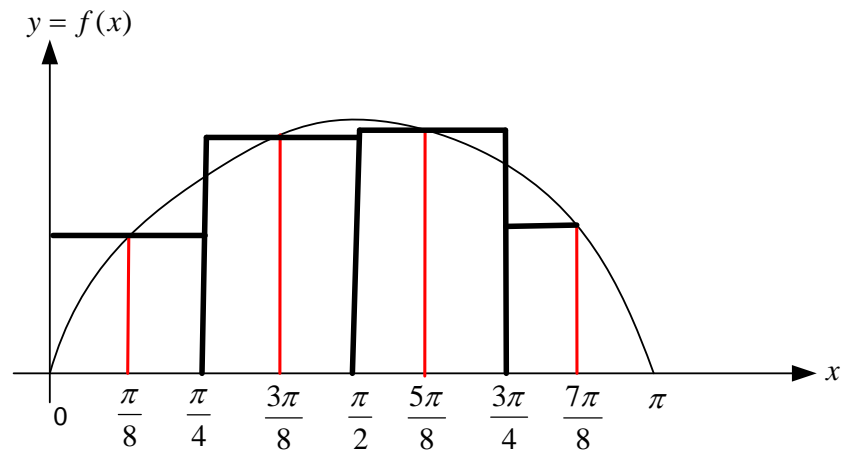
$$h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

$$y = f(x) = \sqrt{x} \, \sin x$$

$$\int_a^b f(x)dx = \int_0^\pi \sqrt{x} \, \sin x \, dx \cong h \left[f\left(\frac{\pi}{8}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{5\pi}{8}\right) + f\left(\frac{7\pi}{8}\right) \right]$$

$$y = f(x) = \sqrt{x} \, \sin x$$

$$f\left(\frac{\pi}{8}\right) = \sqrt{\frac{\pi}{8}} \, \sin\left(\frac{\pi}{8}\right) = 0.626657 \times 0.3826834 = 0.2398112$$

$$f\left(\frac{3\pi}{8}\right) = \sqrt{\frac{3\pi}{8}} \, \sin\left(\frac{3\pi}{8}\right) = 1.0854019 \times 0.9238795 = 1.0027805$$

$$f\left(\frac{5\pi}{8}\right) = \sqrt{\frac{5\pi}{8}} \, \sin\left(\frac{5\pi}{8}\right) = 1.4012478 \times 0.9238795 = 1.2945842$$

$$f\left(\frac{7\pi}{8}\right) = \sqrt{\frac{7\pi}{8}} \, \sin\left(\frac{7\pi}{8}\right) = 1.6579788 \times 0.3826834 = 0.634481$$

$$\int_a^b f(x)dx = \int_0^\pi \sqrt{x} \, \sin x \, dx \cong h \left[f\left(\frac{\pi}{8}\right) + f\left(\frac{3\pi}{8}\right) + f\left(\frac{5\pi}{8}\right) + f\left(\frac{7\pi}{8}\right) \right]$$

$$\int_a^b f(x)dx = \int_0^\pi \sqrt{x} \, \sin x \, dx \cong \frac{\pi}{4}\left[0.2398112 + 1.0027805 + 1.2945842 + 0.634481 \right]$$

$$\int_a^b f(x)dx = \int_0^\pi \sqrt{x} \, \sin x \, dx \cong \frac{\pi}{4}\left[3.1716569 \right] = 2.4910135 \cong 2.49 \ (3 \ s.f)$$

Simpson's Rule

$a = 0, \quad b = \pi$
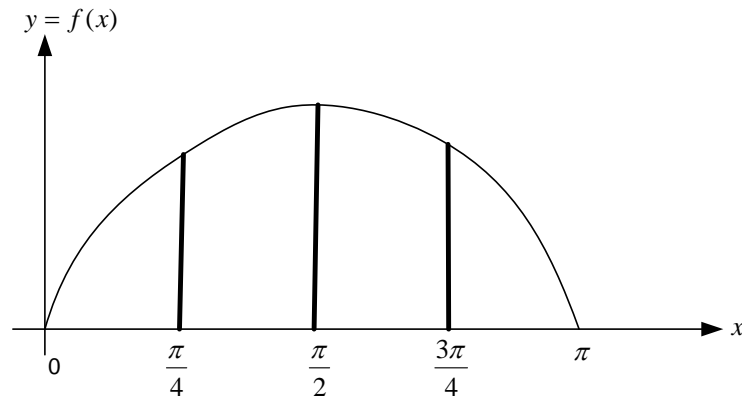
$h = \dfrac{b-a}{n} = \dfrac{\pi-0}{4} = \dfrac{\pi}{4}$

$y = f(x) = \sqrt{x}\ \sin x$

$\displaystyle\int_a^b f(x)dx = \int_0^\pi \sqrt{x}\ \sin x\,dx \cong \dfrac{h}{3}\left[f(a) + f(b) + 4\{f(a+h) + f(a+3h) + ...\} + 2\{f(a+2h) + f(a+4h) + ...\}\right]$

$y = f(x) = \sqrt{x}\ \sin x$

$f(a) = f(0) = \sqrt{x}\ \sin x = \sqrt{0}\ \sin 0 = 0$

$f(a+h) = f\left(0 + \dfrac{\pi}{4}\right) = \sqrt{\dfrac{\pi}{4}}\ \sin\left(\dfrac{\pi}{4}\right) = 0.8862269 \times 0.7071067 = 0.626657$

$f(a+2h) = f\left(0 + \dfrac{2\pi}{4}\right) = f\left(\dfrac{\pi}{2}\right) = \sqrt{\dfrac{\pi}{2}}\ \sin\left(\dfrac{\pi}{2}\right) = 1.2533141 \times 1 = 1.12533141$

$f(a+3h) = f\left(0 + \dfrac{3\pi}{4}\right) = f\left(\dfrac{3\pi}{4}\right) = \sqrt{\dfrac{3\pi}{4}}\ \sin\left(\dfrac{3\pi}{4}\right) = 1.5349901 \times 0.7071067 = 1.0854019$

$f(a+4h) = f\left(0 + \dfrac{4\pi}{4}\right) = f(\pi) = f(b) = \sqrt{\pi}\ \sin(\pi) = 1.7724539 \times 0 = 0$

$\displaystyle\int_a^b f(x)dx = \int_0^\pi \sqrt{x}\ \sin x\,dx \cong \dfrac{h}{3}\left[f(a) + f(b) + 4\{f(a+h) + f(a+3h) + ...\} + 2\{f(a+2h)\}\right]$

$\displaystyle\int_a^b f(x)dx = \int_0^\pi \sqrt{x}\ \sin x\,dx \cong \dfrac{\pi}{4\times 3}\left[0 + 0 + 4\{0.626657 + 1.0854019\} + 2\{1.1253314\}\right] = 2.382086 \cong 2.38\ (3\ s.f)$

PAPER-46

Remainder of a polynomial f(x)

$f(x) = (x-1)(x+2)(x-3) = x^3 - 2x^2 - 5x + 6$

As it can be seen very clearly that $(x-1)$, $(x+2)$ and $(x-3)$ are the factors of $f(x)$. Hence, $f(x) = 0$ when $x = 1, x = -2, x = 3$. So, $f(1) = 0, f(-2) = 0, f(3) = 0$

Note: if $f(x) = (x-1)(x+2)(x-3) = x^3 - 2x^2 - 5x + 6$ is divided by any of its factors, the remainder will be zero.

Ex-46-1: $f(x) = (2x+3)(3x+1)(x-2) = 6x^3 - x^2 - 19x - 6$

As it can be seen very clearly that $(2x+3)$, $(3x+1)$ and $(x-2)$ are the factors of $f(x)$. Hence, $f(x) = 0$ when $x = -\dfrac{3}{2}, x = -\dfrac{1}{3}, x = 2$. So, $f\left(-\dfrac{3}{2}\right) = 0, f\left(-\dfrac{1}{3}\right) = 0, f(2) = 0$

Note: if $f(x) = (2x+3)(3x+1)(x-2) = 6x^3 - x^2 - 19x - 6$ is divided by any of its factors, the remainder will be zero.

Ex-46-2: $f(x) = (3-2x)(3x+1)(x-2) = -6x^3 + 19x^2 - 11x - 6$

As it can be seen very clearly that $(3-2x)$, $(3x+1)$ and $(x-2)$ are the factors of $f(x)$. Hence, $f(x) = 0$ when $x = \dfrac{3}{2}, x = -\dfrac{1}{3}, x = 2$. So, $f\left(\dfrac{3}{2}\right) = 0, f\left(-\dfrac{1}{3}\right) = 0, f(2) = 0$

Note: if $f(x) = (3-2x)(3x+1)(x-2) = -6x^3 + 19x^2 - 11x - 6$ is divided by any of its factors, the remainder will be zero.

Ex-46-3: If $f(x) = -6x^3 + 19x^2 - 11x - 6 + 10 = -6x^3 + 19x^2 - 11x + 4$ is divided by $(3-2x)$, $(3x+1)$, $(x-2)$, the remainder is 10, because

$$f\left(\frac{3}{2}\right) = -6\left(\frac{3}{2}\right)^3 + 19\left(\frac{3}{2}\right)^2 - 11\left(\frac{3}{2}\right) + 4 = -6\left(\frac{27}{8}\right) + 19\left(\frac{3}{2}\right)^2 - 11\left(\frac{3}{2}\right) + 4$$

$$f\left(\frac{3}{2}\right) = -\frac{81}{4} + \frac{171}{4} - \frac{33}{2} + 4 = -\frac{81}{4} + \frac{171}{4} - \frac{66}{4} + \frac{16}{4} = \frac{40}{4} = 10$$

$$f\left(-\frac{1}{3}\right) = -6\left(-\frac{1}{3}\right)^3 + 19\left(-\frac{1}{3}\right)^2 - 11\left(-\frac{1}{3}\right) + 4 = -6\left(-\frac{1}{27}\right) + 19\left(\frac{1}{9}\right) - 11\left(-\frac{1}{3}\right) + 4$$

$$f\left(-\frac{1}{3}\right) = \frac{6}{27} + \frac{19}{9} + \frac{11}{3} + 4 = \frac{54}{9} + 4 = 10$$

$$f(2) = -6(2)^3 + 19(2)^2 - 11(2) + 4 = -48 + 76 - 22 + 4 = 10$$

Note: if
$f(x) = (3 - 2x)(3x + 1)(x - 2) + 10 = -6x^3 + 19x^2 - 11x - 6 + 10 = -6x^3 + 19x^2 - 11x + 4$ is divided by any of its factors, the remainder will be 10 as follows.

The function $f(x) = x^4 + bx + c$ is such that $f(2) = 0$. Also, when $f(x)$ is divided by $x + 3$, the remainder is 85. Find the values of a and b.

Ans: $f(x) = x^4 + bx + c$

$$f(2) = 2^4 + b \times 2 + c = 0 \Rightarrow 2b + c = -2^4 = -16$$

$$2b + c = -16 \qquad \text{...(1)}$$

And $f(-3) = (-3)^4 + b \times (-3) + c = 85 \Rightarrow -3b + c = 85 - (-3)^4$

$$-3b + c = 85 - 81 = 4$$

$$-3b + c = 4 \qquad \text{...(2)}$$

Hence,

$$2b + c = -16 \qquad \text{...(1)}$$

$$-3b + c = 4 \qquad \text{...(2)}$$

Subtracting Eq.2 from Eq.1:

$$2b - (-3b) = -16 - 4$$

$$2b + 3b = -20$$

$$5b = -20$$

$$b = -4$$

$$2b + c = -16 \qquad \text{...(1)}$$

$$2 \times (-4) + c = -16 \Rightarrow -8 + c = -16 \Rightarrow c = -16 + 8 = -8$$

So:

$$b = -4$$

$$c = -8$$

PAPER-47

Some additional practice

Ex-47-1:　You are given that $f(x) = x^4 + ax - 6$ and that $(x - 2)$ is a factor of $f(x)$. Find the value of a.

Ans:　$f(x) = x^4 + ax - 6$

If $x - 2$ **is a factor of** $f(x) = x^4 + ax - 6$, **then** $f(2) = 2^4 + a(2) - 6 = 0$

$\Rightarrow 2a = 6 - 16 = -10 \qquad \Rightarrow a = -5$

Ex-47-2:　When $x^3 + kx + 7$ is divided by $(x - 2)$, the remainder is 3. Find the value of k.

Ans: $f(x) = x^3 + kx + 7 \quad \Rightarrow f(2) = 2^3 + 2k + 7 = 3 \quad \Rightarrow 2k = 3 - 7 - 8 = -12$

$\Rightarrow k = -6$

Ex-47-3: **Consider the function** $f(x) = x^2 + 2x + 1$

$f(-1) = (-1)^2 + 2(-1) + 1 = 1 - 2 + 1 = 2 - 2 = 0$, therefore, if $f(x) = x^2 + 2x + 1$, is divided by $x + 1$, the remainder will be **zero** as shown below.

$$
\begin{array}{r}
x + 1 \\
x + 1 \overline{\smash{\big)}\, x^2 + 2x + 1} \\
\underline{x^2 + x} \\
x + 1 \\
\underline{x + 1} \\
0
\end{array}
$$

Ex-47-4: **Consider the function** $f(x) = x^2 + 2x + 2$

$f(-1) = (-1)^2 + 2(-1) + 2 = 1 - 2 + 2 = 3 - 2 = 1$, therefore, if $f(x) = x^2 + 2x + 2$, is divided by $x + 1$, the remainder will be **1** as shown below.

$$
\begin{array}{r}
x + 1 \\
x + 1 \overline{\smash{\big)}\, x^2 + 2x + 2} \\
\underline{x^2 + x} \\
x + 2 \\
\underline{x + 1} \\
1
\end{array}
$$

Ex-47-5: **Consider the function** $f(x) = x^2 + 2x + 5$

$f(-1) = (-1)^2 + 2(-1) + 5 = 1 - 2 + 5 = 6 - 2 = 4$, therefore, if $f(x) = x^2 + 2x + 5$, is divided by $x + 1$, the remainder will be **4** as shown below.

$$\begin{array}{r}
x+1 \\
x+1 \overline{\smash{\big)}\ x^2+2x+5} \\
x^2+x \\
\hline
x+5 \\
x+1 \\
\hline
4
\end{array}$$

Ex-47-6: **Consider the function** $f(x) = x^4 - 4x - 8$

$f(2) = (2)^4 - 4(2) - 8 = 16 - 8 - 8 = 16 - 16 = 0$, therefore, if $f(x) = x^4 - 4x - 8$, is divided by $x - 2$, the remainder will be **0** as shown below.

$$\begin{array}{r}
x^3 + 2x^2 + 4x + 2 \\
x-2 \overline{\smash{\big)}\ x^4 - 4x - 8} \\
x^4 - 2x^3 \\
\hline
2x^3 - 4x \\
2x^3 - 4x^2 \\
\hline
4x^2 - 4x \\
4x^2 - 8x \\
\hline
4x - 8 \\
4x - 8 \\
\hline
0
\end{array}$$

Ex-47-7:   **The function** $f(x) = x^4 + bx + c$ is such that $f(2) = 0$. Also, when $f(x)$ is divided by $x + 3$, the remainder is 85. Find the values of a and b.

Ans: $f(x) = x^4 + bx + c$

$f(2) = 2^4 + b \times 2 + c = 0 \Rightarrow 2b + c = -2^4 = -16$

$2b + c = -16$ ...(1)

And $f(-3) = (-3)^4 + b \times (-3) + c = 85 \Rightarrow -3b + c = 85 - (-3)^4$

$-3b + c = 85 - 81 = 4$

$-3b + c = 4$ ...(2)

Hence,

$$2b + c = -16 \qquad \text{...(1)}$$

$$-3b + c = 4 \qquad \text{...(2)}$$

Subtracting Eq.2 from Eq.1:

$$2b - (-3b) = -16 - 4$$

$$2b + 3b = -20$$

$$5b = -20$$

$$b = -4$$

$$2b + c = -16 \qquad \text{...(1)}$$

$$2 \times (-4) + c = -16 \Rightarrow -8 + c = -16 \Rightarrow c = -16 + 8 = -8$$

So: $\quad b = -4 \qquad$ and $\qquad c = -8$

PAPER-48

Some additional practice

Ex-48-1: You are given that $f(x) = x^4 + ax - 6$ and that $(x - 2)$ is a factor of $f(x)$. Find the value of a.

Ans: $f(x) = x^4 + ax - 6$

If $x - 2$ **is a factor of** $f(x) = x^4 + ax - 6$**, then** $f(2) = 2^4 + a(2) - 6 = 0$

$$\Rightarrow 2a = 6 - 16 = -10 \qquad \Rightarrow a = -5$$

Ex-48-2: When $x^3 + kx + 7$ is divided by $(x - 2)$, the remainder is 3. Find the value of k.

Ans: $f(x) = x^3 + kx + 7 \qquad \Rightarrow f(2) = 2^3 + 2k + 7 = 3 \Rightarrow 2k = 3 - 7 - 8 = -12$

$$\Rightarrow k = -6$$

Ex-48-3: **Consider the function** $f(x) = x^2 + 2x + 1$

$f(-1) = (-1)^2 + 2(-1) + 1 = 1 - 2 + 1 = 2 - 2 = 0$, therefore, if $f(x) = x^2 + 2x + 1$, is divided by $x + 1$, the remainder will be **zero** as shown below.

$$
\begin{array}{r}
x + 1 \\
\hline
x + 1 \, \big|\, x^2 + 2x + 1 \\
x^2 + x \\
\hline
x + 1 \\
x + 1 \\
\hline
0
\end{array}
$$

Ex-48-4: **Consider the function** $f(x) = x^2 + 2x + 2$

$f(-1) = (-1)^2 + 2(-1) + 2 = 1 - 2 + 2 = 3 - 2 = 1$, therefore, if $f(x) = x^2 + 2x + 2$, is divided by $(x + 1)$, the remainder will be **1** as shown below.

$$
\begin{array}{r}
x + 1 \\
\hline
x + 1 \, \big|\, x^2 + 2x + 2 \\
x^2 + x \\
\hline
x + 2 \\
x + 1 \\
\hline
1
\end{array}
$$

Ex-48-5: **Consider the function** $f(x) = x^2 + 2x + 5$

$f(-1) = (-1)^2 + 2(-1) + 5 = 1 - 2 + 5 = 6 - 2 = 4$, therefore, if $f(x) = x^2 + 2x + 5$, is divided by $(x + 1)$, the remainder will be **4** as shown below.

$$\begin{array}{r} x+1 \\ x+1\,\overline{\big)\,x^2+2x+5} \\ \underline{x^2+x} \\ x+5 \\ \underline{x+1} \\ 4 \end{array}$$

Ex-48-6: **Consider the function** $f(x)=x^4-4x-8$

$f(2)=(2)^4-4(2)-8=16-8-8=16-16=0$, therefore, if $f(x)=x^4-4x-8$, is divided by $(x-2)$, the remainder will be **0** as shown below.

$$\begin{array}{r} x^3+2x^2+4x+2 \\ x-2\,\overline{\big)\,x^4-4x-8} \\ \underline{x^4-2x^3} \\ 2x^3-4x \\ \underline{2x^3-4x^2} \\ 4x^2-4x \\ \underline{4x^2-8x} \\ 4x-8 \\ \underline{4x-8} \\ \mathbf{0} \end{array}$$

Ex-48-7: **The polynomial** $p(x)=x^3-7x-6$

(a)   Use the factor theorem to show that $(x+1)$ is a factor.

(b)   Express $p(x)=x^3-7x-6$ as the product of three linear factors.

If $x+1$ is a factor of $p(x)=x^3-7x-6$, then $p(-1)=(-1)^3-7(-1)-6\overset{?}{=}0$

$\Rightarrow -1+7-6=0 \qquad \Rightarrow OK$

$$\begin{array}{r} x^2 - x - 6 \\ \hline x+1 \overline{\smash{\big)}\ x^3 - 7x - 6} \\ \underline{x^3 + x^2} \\ -x^2 - 7x \\ -x^2 - x \\ \hline -6x - 6 \\ -6x - 6 \\ \hline 0 \end{array}$$

$$p(x) = x^3 - 7x - 6 = (x+1)(x^2 - x - 6) = (x+1)(x-3)(x+2)$$

Ex-48-8:   **The polynomial** $p(x)$ is given by $p(x) = x^3 + x - 10$

(a)   Use the factor theorem to show that $(x-2)$ is a factor of $p(x)$.

(b)   Express $p(x)$ in the form $p(x) = (x-2)(x^2 + ax + b)$, where a and b are constants.

Ans: $p(x) = x^3 + x - 10$
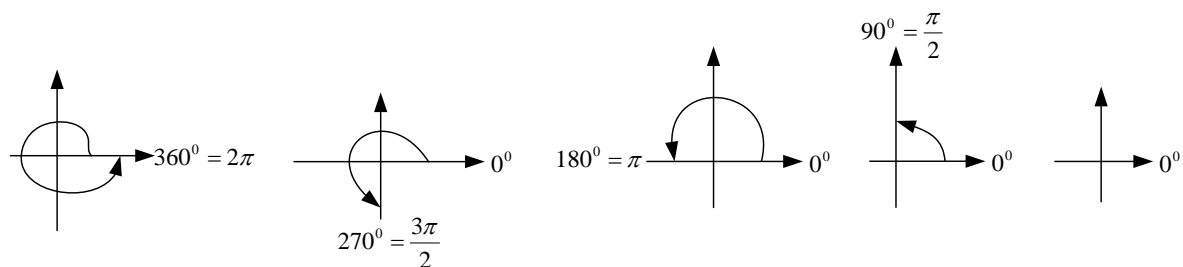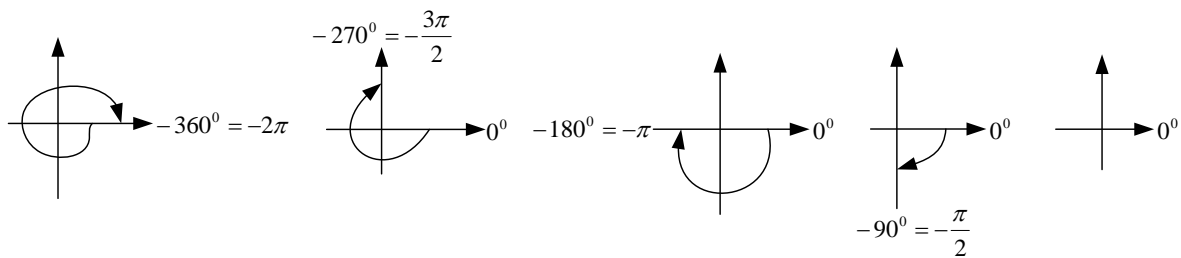
$$p(2) = 2^3 + 2 - 10 = 8 + 2 - 10 = 0$$

Because p(2)=0, then x-2 is a factor of p(x).

PAPER-49

Solving Trigonometric Equations.
Ex-49-1:   Find the solution to the equations in the ranges indicated.

$$-270^0 = -\frac{3\pi}{2}$$
$$-360^0 = -2\pi$$
$$0^0$$
$$-180^0 = -\pi$$
$$0^0$$
$$0^0$$
$$0^0$$
$$-90^0 = -\frac{\pi}{2}$$

Ex-49-2: $2\sin 1.5x = 1.8$ *for* $0^0 \leq x \leq 360^0$

$\sin 1.5x = \dfrac{1.8}{2} = 0.9$

$1.5x = \sin^{-1}(0.9)$

$1.5x = 64.16^0, 115.84^0, 424.16^0, 475.83^0$

$x = \left[\dfrac{64.16}{1.5}, \dfrac{115.84}{1.5}, \dfrac{424.16}{1.5}, \dfrac{475.83}{1.5}\right]$

$x = 42.77, 77.226, 282.77, 317.22$
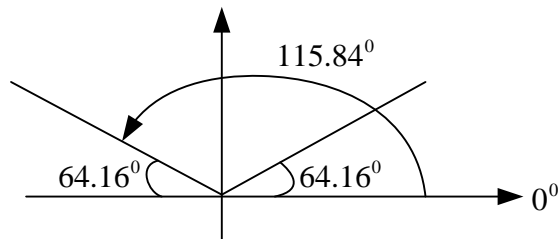
Ex-49-3: $8\cos 16x + 6 = 2$ *for* $0^0 \leq x \leq 50^0$

$8\cos 16x = 2 - 6$

$8\cos 16x = -4$

$\cos 16x = \dfrac{-4}{8} = -0.5$

$16x = \cos^{-1}(-0.5)$

$16x = 120^0, 240^0, 480^0, 600^0$

$x = \dfrac{120}{16} = 7.5^0, \dfrac{240}{16} = 15^0, \dfrac{480}{16} = 30^0, \dfrac{600}{16} = 37.5^0$
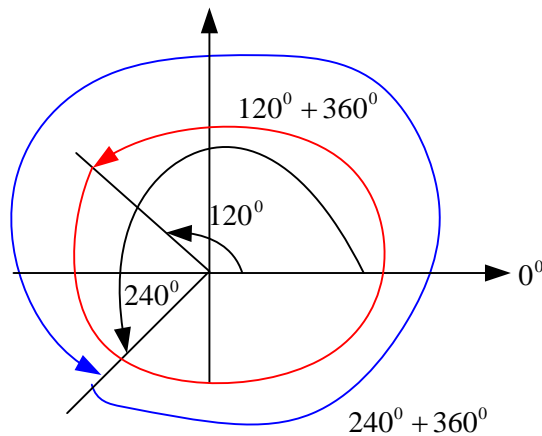
Ex-49-4: $5\sin 60x + 6 = 4$ *for* $0^0 \leq x \leq 30^0$

$5\sin 60x = 4 - 6$

$\sin 60x = -2$

$\sin 60x = \dfrac{-2}{5} = -0.4$

$60x = \sin^{-1}(-0.4)$

$60x = 203.58^0, 336.42^0, 563.58^0, 696.42^0, 923.58^0, 1056.42^0, 1283.58^0, 1416.42^0, 1643.58$

$x = 3.393^0, 5.607^0, 9.393^0, 11.607^0, 15.393^0, 17.607^0, 21.393^0, 27.393^0$

Ex-49-5: $2.5\cos 3x + 12 = 10 \quad for \; -90^0 \le x \le 90^0$

$2.5\cos 3x = 10 - 12$

$2.5\cos 3x = -2$

$\cos 3x = \dfrac{-2}{2.5} = -0.8$

$3x = \cos^{-1}(-0.8)$

$3x = 143.13^0, 216.87^0, -143.13^0, -216.87^0$

$x = 47.71^0, 72.29^0, -47.71^0, -72.29^0$

143.13^0

216.87^0

0^0

-216.87^0

-143.13^0

Ex-49-6: $8\cos\left(\dfrac{x}{2} - 15\right) = 7 \quad for \; 0^0 \le x \le 240^0$

$\cos\left(\dfrac{x}{2} - 15\right) = \dfrac{7}{8}$

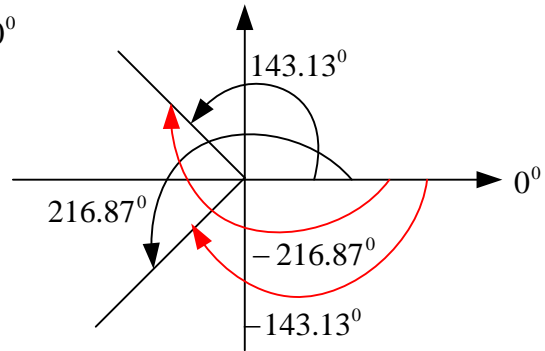$\left(\dfrac{x}{2} - 15\right) = \cos^{-1}\left(\dfrac{7}{8}\right) = 28.96^0$

$\left(\dfrac{x}{2} - 15\right) = 28.96^0, 28.96^0 + 360^0$

$\dfrac{x}{2} = 28.96^0 + 15^0 = 43.96^0 \Rightarrow x = 87.92^0$

Ex-49-7: $-40\cos 12x = 16 \quad for \; 0^0 \le x \le 120^0$

$\cos 12x = \dfrac{-16}{40} = -0.4$

$12x = \cos^{-1}(-0.4) = 113.58^0, \; 246.42^0, \left(113.58^0 + 360^0\right), \left(246.42^0 + 360^0\right), \left(113.58^0 + 720^0\right), \left(246.42^0 + 720^0\right)$
$, \left(113.58^0 + 1080^0\right), \left(246.42^0 + 1080^0\right)$

$x = 9.465^0, 20.535^0, 39.465^0, 50.535^0, 69.465^0, 80.535^0, 99.465^0, 110.535^0$

Ex-49-8: $5 + 6\cos(12x + 8) = 4 \quad for \; 0^0 \le x \le 90^0$

$6\cos(12x + 8) = 4 - 5 = -1$

$(12x + 8) = \cos^{-1}\left(\dfrac{-1}{6}\right) = 99.59^0, \; 260.41^0, 459.59^0, 620.41^0, 819.59^0, 980.41^0$

$12x = 91.59^0, \; 252.41^0, 451.59^0, 612.41^0, 811.59^0, 972.41^0$

$x = 7.325^0, \; 21.034^0, 37.633^0, 51.034^0, 67.633^0, 81.034^0$

PAPER-50

Integration by substitution and by parts

Ex-50-1: Find the integral of $(3-2x)^5$ with respect to x.

$$\int (3-2x)^5 \, dx = ?$$

$$u = 3 - 2x \qquad \Rightarrow \frac{du}{dx} = -2 \qquad \Rightarrow dx = -\frac{du}{2}$$

$$\int (3-2x)^5 \, dx = -\int \frac{1}{2} u^5 \, du = -\frac{1}{2} \int u^5 \, du = -\frac{1}{12} u^6 + c = -\frac{1}{12}(3-2x)^6 + c$$

Ex-50-2: Find $\int_{1}^{2} (1-3x)^3 \, dx =$

$$\int_{1}^{2} (1-3x)^3 \, dx = ?$$

$$u = 1 - 3x \qquad \Rightarrow \frac{du}{dx} = -3 \qquad \Rightarrow dx = -\frac{du}{3}$$

$$\int_{1}^{2} (1-3x)^3 \, dx = -\int \frac{1}{3} u^3 \, du = -\frac{1}{3} \int u^3 \, du = -\frac{1}{12} u^4 = -\frac{1}{12}(1-3x)^4 \Big|_{1}^{2} = -\frac{1}{12}\left[(1-6)^4 - (1-3)^4\right]$$

$$\int_{1}^{2} (1-3x)^3 \, dx = \frac{1}{12}(625 - 16) = 50.75$$

Ex-50-3: Find $\int \frac{x-1}{2x+3} \, dx =$

$$u = 2x + 3 \qquad \Rightarrow \frac{du}{dx} = 2 \qquad \Rightarrow dx = \frac{du}{2}$$

$$u = 2x + 3 \qquad \Rightarrow 2x = u - 3 \qquad \Rightarrow x = \frac{u-3}{2} \qquad \Rightarrow x - 1 = \frac{u-3}{2} - \frac{1}{1} = \frac{u-3-2}{2} = \frac{u-5}{2}$$

$$\int \frac{x-1}{2x+3} \, dx = \int \frac{1}{2} \frac{u-5}{2u} \, du = \frac{1}{4} \int \frac{u-5}{u} \, du = \frac{1}{4} \int \frac{u}{u} \, du - \frac{5}{4} \int \frac{du}{u} = \frac{1}{4} \int du - \frac{5}{4} \int \frac{du}{u} = \frac{1}{4} u - \frac{5}{4} \ln u + c$$

$$\int \frac{x-1}{2x+3} \, dx = \frac{1}{4}(2x+3) - \frac{5}{4} \ln(2x+3) + c$$

Ex-50-4: Find $\int_1^2 \dfrac{x+2}{\sqrt{2x-1}}\,dx =$

$$u = 2x-1 \qquad \Rightarrow \frac{du}{dx} = 2 \quad \Rightarrow dx = \frac{du}{2}$$

$$u = 2x-1 \qquad \Rightarrow 2x = u+1 \quad \Rightarrow x = \frac{u+1}{2} \quad \Rightarrow x+2 = \frac{u+1+4}{2} = \frac{u+5}{2}$$

$$\int_1^2 \frac{x+2}{\sqrt{2x-1}} = \int \frac{1}{2}\frac{u+5}{2\sqrt{u}}\,du = \frac{1}{4}\int \frac{u+5}{\sqrt{u}}\,du = \frac{1}{4}\int \frac{u}{\sqrt{u}}\,du + \frac{5}{4}\int \frac{du}{\sqrt{u}} = \frac{1}{4}\int u^{\frac{1}{2}}\,du + \frac{5}{4}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{4}\left(\frac{2}{3}\right)u^{\frac{3}{2}} + \frac{5}{4}\left(\frac{1}{2}\right)u^{\frac{1}{2}} = \frac{1}{6}u^{\frac{3}{2}} + \frac{5}{8}u^{\frac{1}{2}}$$

$$\int_1^2 \frac{x+2}{\sqrt{2x-1}}\,dx = \frac{1}{6}(2x-1)^{\frac{3}{2}}\Big|_1^2 + \frac{5}{8}(2x-1)^{\frac{1}{2}}\Big|_1^2 = \frac{1}{6}\left[\sqrt{27}-1\right] + \frac{5}{8}[3-1] = \frac{1}{6}\left[\sqrt{27}-1\right] + \frac{15}{8}$$

$$\int_1^2 \frac{x}{2x+3}\,dx = \frac{1}{6}\left[3\sqrt{3}-1\right] + \frac{15}{8} = \frac{\sqrt{3}}{2} - \frac{1}{6} + \frac{15}{8} = \frac{\sqrt{3}}{2} + \frac{21}{12} = \frac{\sqrt{3}}{2} + \frac{7}{4}$$

Ex-50-5: Find $\int \dfrac{1}{\sqrt{3x-1}}\,dx =$

$$u = 3x-1 \qquad \Rightarrow \frac{du}{dx} = 3 \quad \Rightarrow dx = \frac{du}{3}$$

$$\int \frac{1}{3}\frac{du}{\sqrt{u}} = \frac{1}{3}\int \frac{du}{\sqrt{u}} = \frac{1}{3}\int u^{-\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{1}{2}} = \frac{2}{3}\sqrt{u} = \frac{2}{3}\sqrt{2x-1}$$

Ex-50-6: Find $\int \dfrac{x^2}{\left(x^3-1\right)^2}\,dx = ?$

$$u = x^3 - 1 \qquad \Rightarrow \frac{du}{dx} = 3x^2 \qquad \Rightarrow dx = \frac{du}{3x^2}$$

$$\int \frac{x^2}{\left(x^3-1\right)^2}\,dx = \int \frac{x^2\,du}{3x^2 u^2} = \frac{1}{3}\int \frac{du}{u^2} = \frac{1}{3}\int u^{-2}\,du = -\frac{1}{3}u^{-1} = -\frac{1}{3(x^3-1)} + c$$

Ex-50-7: Given that $\dfrac{dy}{dx} = \sqrt{2x-1}$ and $y = 5$ when $x = 0$, find y.

$$\frac{dy}{dx} = \sqrt{2x-1}$$

$$dy = \sqrt{2x-1}\,dx \qquad \Rightarrow \int dy = \int \sqrt{2x-1}\,dx$$

$$y = \int \sqrt{2x-1}\, dx$$

$$u = 2x - 1 \qquad \Rightarrow \frac{du}{dx} = 2 \qquad \Rightarrow dx = \frac{du}{2}$$

$$y = \int \sqrt{2x-1}\, dx = \frac{1}{2} \int \sqrt{u}\, du = \frac{1}{2} \int u^{\frac{1}{2}}\, du = \frac{1}{2} \left(\frac{2}{3}\right) u^{\frac{3}{2}} = \frac{1}{3}(2x-1)^{\frac{3}{2}} + c$$

$$y = \frac{1}{3}(1-2x)^{\frac{3}{2}} + c \qquad 5 = 0 + c \qquad \Rightarrow c = 5$$

$$y = \frac{1}{3}(2x-1)^{\frac{3}{2}} + 5$$

Ex-50-8: Find $\displaystyle\int_0^4 \frac{x-4}{\sqrt{4-x}}\, dx =$

$$u = x - 4 \qquad \Rightarrow \frac{du}{dx} = 1 \qquad \Rightarrow dx = du$$

$$u = x - 4 \qquad \Rightarrow x = 4 + u$$

$$\int_0^4 \frac{x-4}{\sqrt{4-x}}\, dx = -\int \frac{4+u-4}{\sqrt{u}}\, du = \int \frac{u}{\sqrt{u}}\, du = \int u^{\frac{1}{2}}\, du$$

$$= \frac{2}{3} u^{\frac{3}{2}} = \frac{2}{3}(4-x)^{\frac{3}{2}} \Big|_0^4 = \frac{2}{3}(0-8) = -\frac{16}{3}$$

Ex-50-9: Find $\displaystyle\int_0^{\frac{1}{3}} \frac{1}{\sqrt{3x+1}}\, dx =$

$$u = 3x + 1 \qquad \Rightarrow \frac{du}{dx} = 3 \qquad \Rightarrow dx = \frac{du}{3}$$

$$\int_0^{\frac{1}{3}} \frac{1}{\sqrt{3x+1}}\, dx = \int \frac{1}{3} \frac{du}{\sqrt{u}} = \frac{1}{3} \int \frac{du}{\sqrt{u}} = \frac{1}{3} \int u^{-\frac{1}{2}}\, du = \frac{2}{3} u^{\frac{1}{2}} = \frac{2}{3} \sqrt{u} = \frac{2}{3} \sqrt{3x+1} \Big|_0^{\frac{1}{3}}$$

$$\int_0^{\frac{1}{3}} \frac{1}{\sqrt{3x+1}}\, dx = \frac{2}{3} \sqrt{3x+1} \Big|_0^{\frac{1}{3}} = \frac{2}{3}\left(\sqrt{2} - 1\right)$$

Integration by Parts

The formula for integration by parts is: $\int u \dfrac{dv}{dx} dx = uv - \int v \dfrac{du}{dx} dx$

Ex-50-10: $\int x e^{-x} dx =$

$\int x e^{3x} dx = \int u \, dv = uv - \int v \, du \quad \text{int } egration \ by \ parts$

$Let \quad u = x \quad and \quad dv = e^{-x} dx$

$du = dx \quad and \quad v = \int e^{-x} dx = -e^{-x}$

$\int x e^{-x} dx = \int u \, dv = uv - \int v \, du = -xe^{-x} - \int -e^{-x} dx = -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x} + C.$

Ex-50-11: $\int x \cos 2x \, dx =$

$\int x \cos 2x \, dx = \int u \, dv = uv - \int v \, du \quad \text{int } egration \ by \ parts.$

$Let \quad u = x \quad and \quad dv = \cos 2x \, dx$

$du = dx \quad and \quad v = \int \cos 2x \, dx = \dfrac{1}{2} \sin 2x$

$\int x \cos 2x \, dx = \int u \, dv = uv - \int v \, du = \dfrac{x}{2} \sin 2x - \int \dfrac{1}{2} \sin 2x \, dx = \dfrac{x}{2} \sin 2x + \dfrac{1}{4} \cos 2x + C.$

Ex-50-12: $\int (2x+1) \ln x \, dx =$

$\int (2x+1) \ln x \, dx = \int u \, dv = uv - \int v \, du \quad \text{int } egration \ by \ parts.$

$Let \quad u = \ln x \quad and \quad dv = (2x+1) dx$

$du = \dfrac{1}{x} dx \quad and \quad v = \int (2x+1) dx = x^2 + x$

$\int (2x+1) \ln x \, dx = \int u \, dv = uv - \int v \, du = (x^2 + x) \ln x - \int \left(\dfrac{x^2 + x}{x} \right) dx = (x^2 + x) \ln x - \int (x+1) dx$

$\int (2x+1) \ln x \, dx = (x^2 + x) \ln x - \int (x+1) dx = (x^2 + x) \ln x - \dfrac{x^2}{2} - x + C$

Ex-50-13: Using integration by parts, $\int x \sqrt{1+2x} \, dx =$

$\int x \sqrt{1+2x} \, dx = \int u \, dv = uv - \int v \, du \quad \text{int } egration \ by \ parts.$

$Let \quad u = x \quad and \quad dv = \sqrt{1+2x} \, dx = (1+2x)^{\frac{1}{2}} dx$

$du = dx \quad and \quad v = \int (1+2x)^{\frac{1}{2}} dx = \dfrac{1}{2} \left(\dfrac{2}{3} \right) (1+2x)^{\frac{3}{2}} = \dfrac{1}{3} (1+2x)^{\frac{3}{2}}$

$$\int x\sqrt{1+2x}\,dx = \int u\,dv = uv - \int v\,du = \frac{x}{3}(1+2x)^{\frac{3}{2}} - \frac{1}{3}\int (1+2x)^{\frac{3}{2}}\,dx = \frac{x}{3}(1+2x)^{\frac{3}{2}} - \frac{1}{6}\left(\frac{2}{5}\right)(1+2x)^{\frac{5}{2}} + C$$

$$\int x\sqrt{1+2x}\,dx = \frac{x}{3}(1+2x)^{\frac{3}{2}} - \frac{1}{15}(1+2x)^{\frac{5}{2}} + C$$

Ex-50-14: $\displaystyle\int_{0}^{2}(1-x)e^{2x}\,dx =$

$$\int_{0}^{2}(1-x)e^{2x}\,dx = \int u\,dv = uv - \int v\,du \quad \text{int } egration\ by\ parts.$$

$$Let \quad u = 1-x \quad and \quad dv = e^{2x}\,dx$$

$$du = -dx \quad and \quad v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$$

$$\int_{0}^{2}(1-x)e^{2x}\,dx = \int u\,dv = uv - \int v\,du = (1-x)\left(\frac{1}{2}\right)e^{2x} - \frac{1}{2}\int e^{2x}(-dx) = \frac{1}{2}(1-x)e^{2x}\Big|_{0}^{2} - \frac{1}{4}e^{2x}\Big|_{0}^{2}$$

$$\int_{0}^{2}(1-x)e^{2x}\,dx = \frac{1}{2}(1-x)e^{2x}\Big|_{0}^{2} - \frac{1}{4}e^{2x}\Big|_{0}^{2} = \frac{1}{2}\left[(1-2)e^{4} - (1-0)\right] - \frac{1}{4}\left[e^{4} - 1\right] = -\frac{1}{2}e^{4} - \frac{1}{4}e^{4} + \frac{1}{4} = -\frac{3}{4}e^{4} - \frac{1}{4}$$

Ex-50-15: $\displaystyle\int_{1}^{2}x^{3}\ln x\,dx =$

$$\int_{1}^{2}x^{3}\ln x\,dx = \int u\,dv = uv - \int v\,du \quad \text{int } egration\ by\ parts.$$

$$Let \quad u = \ln x \quad and \quad dv = x^{3}\,dx$$

$$du = \frac{1}{x}\,dx \quad and \quad v = \int x^{3}\,dx = \frac{x^{4}}{4}$$

$$\int_{1}^{2}x^{3}\ln x\,dx = \int u\,dv = uv - \int v\,du = \ln x\left(\frac{x^{4}}{4}\right) - \frac{1}{4}\int \frac{1}{x}x^{4}\,dx. = \frac{x^{4}}{4}\ln x\Big|_{1}^{2} - \frac{1}{16}x^{4}\Big|_{1}^{2} = 4\ln 2 - 0 - 1 + \frac{1}{16}$$

$$= 4\ln 2 - \frac{15}{16}$$

Ex-50-16: $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x\cos x\,dx =$

$$\int u\,dv = uv - \int v\,du \quad \text{int } egration\ by\ parts.$$

$$Let \quad u = x \quad and \quad dv = \cos x\,dx$$

$$du = dx \quad\quad and \quad\quad v = \int \cos x dx = \sin x$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x dx = \int u dv = uv - \int v du = x \sin x - \int \sin x dx = x \sin x \left.\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \cos x \left.\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}.$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x \cos x dx = \frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\right) \sin\left(-\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) - \cos\left(-\frac{\pi}{2}\right) = \frac{\pi}{2} - \frac{\pi}{2} + 0 - 0 = 0$$

Ex-50-17: The area under the graph of $y = \ln x$ between x=1 and x=5 is given by

$$\int_{1}^{5} \ln x dx = ?$$

$$\int_{1}^{5} \ln x dx = \int u dv = uv - \int v du \quad\quad int\,egration\ by\ parts.$$

$$Let \quad u = \ln x \quad\quad and \quad\quad dv = dx$$

$$du = \frac{1}{x} dx \quad\quad and \quad\quad v = \int dx = x$$

$$\int_{1}^{5} \ln x dx = \int u dv = uv - \int v du = x \ln x - \int \frac{x}{x} dx. = x \ln x \left.\right|_{1}^{5} - x \left.\right|_{1}^{5} = 5 \ln 5 - \ln 1 - (5 - 1) = 5 \ln 5 - 4$$

Ex-50-18: The area under the graph of $y = x \sin x$ and the x axis between x=0 and $x = 2\pi$ is given by

$$\int_{0}^{2\pi} x \sin x dx = ?$$

$$\int_{0}^{2\pi} x \sin x dx = \int u dv = uv - \int v du \quad\quad int\,egration\ by\ parts.$$

$$Let \quad u = x \quad\quad and \quad\quad dv = \sin x dx$$

$$du = dx \quad\quad and \quad\quad v = \int \sin dx = -\cos x$$

$$\int_{0}^{2\pi} x \sin x dx = \int u dv = uv - \int v du = -x \cos x - \int (-\cos x) dx. = -x \cos x \left.\right|_{0}^{2\pi} + \sin x \left.\right|_{0}^{2\pi}$$

$$\int_{0}^{2\pi} x \sin x dx = -x \cos x \left.\right|_{0}^{2\pi} + \sin x \left.\right|_{0}^{2\pi} = -2\pi \cos(2\pi) - 0 + \sin(2\pi) - \sin(0) = -2\pi$$

Dear Brothers and Sisters! السلام عليكم و رحمت الله و بركاته

I kindly request each one of you to remember me, my parents, my children, my family and the entire Muslims of the world in your daily Prayers. **If you have not done it so far, then please do it now.**

ګرانو او محترمو ورونو اوخویندو السلام علیکم ورحمت الله وبرکاته !

له الله(ج) څخه هیله کوم چه تاسي ټول روغ او جوړ وی او الله(ج) د وکړي چي تاسي ما ته، زما مور او پلار ته، زما اولادونو ته، زما ټولي کورنۍ اوټولو مسلمانانو ته د زړه له کومي دعا کړیبي وي، **اوکنه؟** نو اوس یي لطفاً وکړی! والسلام. عبدالله وردک. **زما اریکه :** abdullahwardak53@gmail.com

په پای کې له تاسو ټولو سره له لرې ځایه خدای په اماني کوم. له الله(ج) څخه هیله کوم چی په افغانستان او ټوله نړی کې هغه شانته صلح راشي چی الله(ج) ته مقبوله او منظوره وي. آمین، یارب العالمین.